# Weighted index of graph efficiency improvements for a by-production technology and its application to Indian coal-based thermal power sector.

Sushama Murty[†*] and Resham Nagpal[††*]

September 10, 2018

*Centre for International Trade and Development, School of International Studies, Jawaharlal Nehru University, New Delhi 110067.

Emails: † sushama.murty1@gmail.com †† reshamnagpal89@gmail.com

**Abstract**

In contrast to conventional output-based efficiency indexes that hold input-levels fixed, a graph index of efficiency-improvements (EIs) is derived for a by-production technology by optimizing a weighted average of EIs in input and good and bad output-directions. Trade-offs, which determine optimal-EIs, arise between EIs in good and bad outputs when inefficiencies are removed in emission-causing input-directions. The optimal configurations of EIs for Indian coal-based thermal power plants depend on weights assigned and are correlated with output-based productive-efficiency (OBPE). EIs for plants with high OBPE is limited. With equal weights assigned to EIs in both outputs, optima of plants with moderate OBPE involve greatest EIs in coal-usage and $CO_2$-generation, with no electricity-expansion, while most plants with low OBPE need focus only on electricity-expansion with existing coal-usage. With increasing weight on $CO_2$-reduction, EIs in coal-usage and emission-generation at existing electricity-levels become optimal for increasingly more plants, EIs being greatest for lowest-OBPE plants.

*JEL classification codes:* Q50, Q40, D24,

*Keywords:* emission-generating technologies; by-production technologies; output-based productive, environmental, and overall technical efficiency indexes; weighted graph efficiency indexes; efficiency improvements,

# Weighted index of graph efficiency improvements for a by-production technology and its application to Indian coal-based thermal power sector.

## 1 Introduction.

For a producing unit that generates harmful emissions, improvements in technical efficiency entail, *ceteris-paribus*, increases in the production of its intended/good/desirable output or decreases in its generation of harmful emissions/bad outputs or decreases in the usage of its inputs or some combinations of these.[1] Starting from an inefficient production point, there can be many configurations of efficiency improvements; *e.g.,* a configuration may involve far more efficiency-improvement (increase) in the production of the intended output than efficiency-improvement (reduction) in the generation of harmful emissions, while the reverse may be true in another configuration. A configuration may achieve technical efficiency improvements in production of the good and bad outputs with no change in inputs, while another may, in addition, also involve reductions in usage of some or all inputs. Selection of an optimal configuration of efficiency improvements depends on the choice of an objective function that aggregates over efficiency improvements in all the input and output directions. Optimising such an objective function yields a graph index of efficiency improvements.[2] In this paper, we develop such an index obtained by maximising an objective function that is a weighted average of efficiency improvements in the input and output directions, where the weights can exogenously be chosen by a policy maker or the researcher.[3] We focus on the nature of the technical efficiency improvements that its computation entails, *i.e.,* on the qualitative features of the solution vector of the optimisation problem that computes the weighted graph index of efficiency improvements.

The first step for computing such an index is a specification of a model of an emission-generating technology, relative to which efficiency improvements will be measured. In the literature, three such specifications include the input approach, the weak-disposability based output approach, and the by-production approach.[4] A comparison of the three approaches can

[1]A more rigorous definition will be provided later. More precisely, efficiency improvements are defined as proportional reductions in emission and input-levels and proportional increases in intended output production.

[2]See Färe et al (1985) pp153-154 for a technical efficiency index defined in the full space of inputs and outputs in production models that exclude bad outputs.

[3]In their survey chapter, Murty and Russell (2017) propose a theoretical extension of Färe et al (1985) graph index of technical efficiency to the full space of inputs and good and bad outputs.

[4]See classic works of Baumol and Oates (1988) and Cropper and Oats (1992) for the input approach. It was employed in several works such as Reinhard et al. (1999, 2000), Lee et al. (2002), and Hailu and Veeman (2001). The weak-disposability based output approach can be traced back to Färe et al (1989). It has been

be found in works such as Murty, Russell, and Levkoff (2002, 2012) (henceforth, MRL), Murty (2015), and Murty and Russell (2016, 2017), which advocate the use of the by-production approach.[5] In this work, which focuses on coal-based thermal power plants that produce electricity as the intended output and generate $CO_2$ emission as the bad output, a by-production technology is defined as an intersection of two sub-technologies, denoted by $T_1$ and $T_2$. Sub-technology $T_1$ captures standard neo-classical production relations between inputs of coal-based thermal power plants and the intended output electricity, while sub-technology $T_2$ captures the relation that exists in nature between $CO_2$ emission and combustion of coal, the emission-causing input of coal-based thermal power plants.[6] The (weakly) efficient frontier of sub-technology $T_1$ identifies the maximum amount of electricity that can be produced by this sub-technology for every fixed configuration of its inputs. On the other hand, the lower frontier of sub-technology $T_2$ identifies the minimum level of $CO_2$ emission that can be generated for every fixed amount of the coal input. In MRL, *productive technical efficiency* depends on the proximity of the production point to the efficient frontier of $T_1$, while *environmental technical efficiency* depends on the proximity of the production point to the lower frontier of $T_2$.

In their work, MRL compute the output-based FGL index of technical efficiency. An output-based (in)efficiency index holds levels of inputs fixed and studies efficiency improvements only in the directions of the good and bad outputs.[7] When the FGL index of MRL was employed to compute technical efficiency of Indian coal-based thermal power sector, Murty and Nagpal (2018) found that, while the output-based productive efficiency varies considerably across Indian thermal power plants, the output-based measure of environmental efficiency takes values very close to one for all the plants and is almost perfectly correlated with the usage of aggregate coal input measured in heat units.[8] The latter was attributed to the facts that (i) the data provided by the Central Electricity Authority of India (CEA) on $CO_2$ emission is computed by using a linear deterministic formula that employs an average emission factor, which is constant across

employed in many works such as Coggins and Swinton (1996), Murty and Kumar (2002, 2003) and Sahoo et al. (2017). See Zhou and Poh (2008) for a comprehensive survey.

[5]See also Førsund (1998, 2009, 2017) for a critique of the input and weak-disposability based output approaches. See Serra et al. (2016), Malikov et al. (2015), Kumbhakar and Tsionas (2016), and Ray et al. (2017) for extensions and empirical applications of the by-production approach.

[6]A more general formulation of a by-production technology is one where an emission-generating technology is obtained as an intersection of two or more sub-technologies each capturing distinct production relations between inputs and outputs. See Murty and Russell (2017).

[7]Most indexes for measuring technical efficiency in the presence of bad outputs are output-oriented. Apart from the FGL index, these include the hyperbolic and directional distance function-based efficiency indexes. For those that are developed in the context of the weak-disposability based models of emission-generating technologies, see *e.g.,* Färe et al (1989) and Färe et al (2005).

[8]An aggregate input of coal measured in heat units can be obtained from adding the heat contents of various grades of coal, which are in turn derived from their gross calorific values and the amounts employed. Murty and Nagpal (2018) argue on the appropriateness of measuring the aggregate coal input in heat units rather than in mass units as the former is more consistent with theoretical modelling of emission-generating technologies.

all grades of coal and across all coal-based thermal power plants and (ii) the proportion of the secondary fuel oil employed in these plants is very tiny.[9] Under the assumption of constant returns to scale, Murty and Nagpal (2018) found that the estimated slope of the lower frontier of sub-technology $T_2$ is approximately equal to the constant average emission factor employed by the CEA.[10] Thus, the CEA's method for computing $CO_2$ emission implies that all the thermal power plants operate very close to this frontier, and that there is extremely little scope for efficiency improvement in the environmental direction when the coal input is held fixed.

Murty and Nagpal (2018) conjecture that significant environmental efficiency improvements may actually be possible in the Indian coal-based thermal power sector, when efficiency improvements in coal usage are also modelled and studied; *i.e.,* it is possible that coal-based thermal power plants are wastefully using the coal input during the production of their intended output electricity, and hence are generating excessive amounts of $CO_2$ emission.[11] While the graph index of efficiency improvements that is developed in this paper can be computed for all datasets that include emission generation, the findings and the conjecture in Murty and Nagpal (2018) makes its computation particularly relevant for CEA-type datasets for which no meaningful output-based measures of environmental efficiency can be computed. Thus, in this paper, we ask the question whether there can be significant improvements in environmental and productive efficiency across plants in the Indian coal-based thermal power sector, when the emission-causing input is also allowed to adjust efficiently along with both the good and the bad outputs.

We show that the by-production specification of the technology implies that efficiency improvements in the directions of the good and bad outputs are functions of the efficiency improvements (reductions) in inputs.[12] In fact there is a trade-off between efficiency improvements in the good output and the bad output directions, when inefficiency is sought to be reduced in the coal direction: as coal is reduced, the (maximum) efficiency improvement (*i.e.,* proportionate increase) in electricity production decreases, while the (maximum) efficiency improvement (*i.e.,* proportionate decrease) in $CO_2$ generation increases. To the best of our knowledge, such trade-offs cannot be captured and studied by employing other existing approaches to modelling emission-generating technologies. Thus, the by-production approach captures the costs

[9]The emission factor of a fossil fuel is the amount of emission that is generated per unit heat generated by that fuel.

[10]Under constant returns to scale (which can be argued to be justified by point (i)), the lower frontier of sub-technology $T_2$ is a linear relation between emission and the aggregate input of coal measured in heat units.

[11]Baumgärtner and Arons (2003) illustrate cases where thermodynamic inefficiencies can cause producing units to employ more than the minimal amounts of fossil-fuels needed to produce given amounts of industrial outputs. This would imply generation of more than the minimal amounts of emissions while producing such outputs.

[12]In particular, efficiency improvement in the bad output is a function of the extent of efficiency improvement in the emission-causing input.

and benefits of efficiency improvements in the coal direction, and the optimal configuration of efficiency improvements depends crucially on the weights that the objective function attaches to efficiency improvements in the input, good output, and bad output-directions. Different weighting schemes lead to different solution configurations.

Our empirical results on graph efficiency improvements (which allow inputs to change) are correlated to the output-based measure of productive efficiency computed in Murty and Nagpal (2018) for thermal power plants in our dataset.[13] We classify plants as high, moderate, and low performers based on their output-based productive efficiency and find that optimal efficiency improvements derived while computing the weighted index of graph efficiency improvements vary across these three categories in a more-or-less systematic manner.

When equal and exhaustive weights are assigned to the good and the bad outputs, we find that (i) for a majority of high performers or plants that are placed closest to the efficient frontier of sub-technology $T_1$, the optimum does not involve significant efficiency improvements in the usage of coal and, hence, in the generation of emission, and, in fact, it implies no efficiency improvement in the production of electricity. (ii) for a majority of the moderate performers or plants that are placed farther away from the efficient frontier of sub-technology $T_1$, the optimum continues to recommend (as in the case of majority of the high performers) reductions in the usage of the coal input with no change in electricity generation. But this time, the recommended reductions in the usage of the coal input are significant, which also imply significant reductions in $CO_2$ emission. In fact, plants under this category show maximum potential for reducing $CO_2$ emission when equal weights are assigned to good and bad outputs. (iii) the plants with very low output-based productive efficiency have the greatest potential for increasing electricity generation with their existing levels of usage of the coal input. For these plants, the gain from increase in electricity generation keeping coal unchanged outweighs the gains from emission reduction that could have been achieved by reducing the coal input. (iv) there there are only a few plants in our dataset for which the optimum recommends non-zero efficiency improvements along all three – electricity, emission, and coal – directions, when equal weights are assigned to good and bad outputs.[14]

As we move to weighting schemes that attach more and more weight to the bad output $CO_2$ and less and less weight to the good output electricity, we find that the number of plants for which the optimum recommends efficiency improvements in the coal input direction increases. When full weight is assigned to emission reduction, then for *all* plants the optimum recom-

[13] As mentioned above, output-based productive efficiency index is computed relative to sub-technology $T_1$ holding inputs fixed.

[14] In contrast, points (i) to (iii) above imply that, for a majority of plants, the optimum recommends either efficiency improvement only in the coal (and hence also emission) direction or efficiency improvement only in the electricity direction.

mends minimising usage of the coal input and operating on the isoquants corresponding to their existing levels of electricity generation. Hence, it also recommends maximum possible proportional reductions in emission of $CO_2$, a large part of which is attributed to reductions in coal usage. It also turns out that, while the low performers showed maximum potential for electricity-expansion when equal weights were assigned to efficiency improvements in emission and electricity directions, they switch over to becoming the plants with the highest potential to proportionately reduce coal-usage and hence $CO_2$ emission when the weight on emission increases by more than a half.

In Section 2, we offer a slightly stronger definition of a by-production technology as compared to MRL. Section 3 defines a weighted graph index of efficiency improvements and compares and relates it to the purely output-based FGL measure of technical efficiency developed in MRL. Section 4 relates efficiency improvements in the input directions to efficiency improvements in the directions of good and bad outputs. Section 5 derives the first-order conditions of the optimisation problem defined in Section 3 that computes the weighted index of graph efficiency improvements. It then interprets the first-order conditions in terms of the marginal costs and benefits of efficiency improvements in the input directions. It distinguishes between corner and interior optima. Section 6 constructs Table 1 to showcase all the possible solutions to the optimisation problem set up in Section 3 to compute the weighted index of graph efficiency improvements. Section 8 studies the relations between technical efficiency improvements studied in this paper and the engineering priorities of increasing thermodynamic efficiency and minimising emission per unit of electricity output. Section 9 provides information on the data and methodology employed in our empirical analysis. Employing Table 1 developed in Section 6, Section 10 presents our results and interprets them. We conclude in Section 11. All proofs including tables with plant-level details can be found in the Appendix.

# 2 By-production technologies.

There is one marketed/good/intended/economic output (electricity generated), one bad output ($CO_2$ emission), and $n$ inputs of which the first $n_o$ are non-emission causing, while the remaining $n - n_o = n_z > 0$ are emission-causing. A production vector is denoted by $\langle x, y, z \rangle = \langle x_o, x_z, y, z \rangle \in \mathbf{R}_+^{n+2}$, where $x = \langle x_o, x_z \rangle \in \mathbf{R}_+^n$ and $x_o$ and $x_z$ denote respectively the quantities of non-emission causing and emission-causing inputs used, $y$ is the net output of electricity, and $z$ is the quantity of $CO_2$ emission generated. For $i = 1, \ldots, n_o$, the $i^{th}$ non-emission causing input quantity is denoted by $x_{o_i}$, while for $i = 1, \ldots, n_z$, the $i^{th}$ emission-causing input quantity is denoted by $x_{z_i}$. Depending on convenience, we sometimes also index inputs simply by $i = 1, \ldots, n$, so that $x_i$ denoted the amount of the $i^{th}$ input.

## 2.1 Some concepts for defining by-production technologies.

In this section, we introduce the concept of a costly disposal hull and the lower frontier of a set. In Section 2.2, these will be employed to define a by-production technology.

For any subset $A$ of $\mathbf{R}_+^{n+2}$, define the *costly disposal hull of* $A$ as the set[15]

$$CDH(A) := \left\{ \langle x_o, x_z, y, z \rangle \in \mathbf{R}_+^{n+2} \;\middle|\; \exists \; \langle x_o, x_z', y, z' \rangle \in A \;\text{ such that }\; x_z \leq x_z' \;\text{ and }\; z \geq z' \right\}.$$

From its definition, the costly disposal hull $CDH(A)$ includes set $A$ and all production vectors that contain arbitrarily larger amounts of the emission and arbitrarily lower amounts of the emission-causing inputs than those in $A$. In Figure 1, assume that $n = n_z = 1$ and that set $A$ is denoted by $T_2$.[16] While the upper panels (b) and (c) are examples of set $A$, the lower panels (e) and (f) are their respective costly disposal hulls.[17]

For any subset $A$ of $\mathbf{R}_+^{n+2}$, define the minimum level of feasible emission as:

$$G\left(x_o, x_z, y; A\right) := \inf\{z \geq 0 \mid \langle x_o, x_z, y, z \rangle \in A\}.$$

As will be seen starting from Section 2.2, in this paper, we will mainly be interested in studying the minimum level of emission generated when emission generation is independent of (or does not depend on) the amounts of intended output produced and the non-emission generating inputs used. Hence, in this context, we can ignore variables $y$ and $x_o$ as arguments of function $G$, and write its image solely as

$$z = G\left(x_z; A\right).$$

Then the lower frontier of set $A$ is defined as the set of points that lie on the graph of function $G\left(x_z; A\right)$, namely, it is the set

$$LFront(A) := \left\{ \langle x_o, x_z, y, z \rangle \in \mathbf{R}_+^{n+2} \;\middle|\; z = G\left(x_z; A\right) \right\}.$$

In panel (b) of Figure 1, it can be seen that set $T_2$ and its lower frontier coincide, while this is not so in panel (c), where for every level of input, there exists both a lower and upper bound for emission generation under set $T_2$.

---

[15] Vector notation: Given two vectors $a$ and $b$ in $\mathbf{R}^n$,

$$\begin{aligned} a &\geq b \iff a_i \geq b_i \; \forall \; i = 1, \ldots, n \\ a &> b \iff a \neq b \text{ and } a_i \geq b_i \; \forall \; i = 1, \ldots, n \\ a &\gg b \iff a_i > b_i \; \forall \; i = 1, \ldots, n \end{aligned}$$

[16] Note, Figure 1 depicts only the projection of set $A = T_2$ into the space of the emission and the single input.

[17] See also Murty and Nagpal (2018).

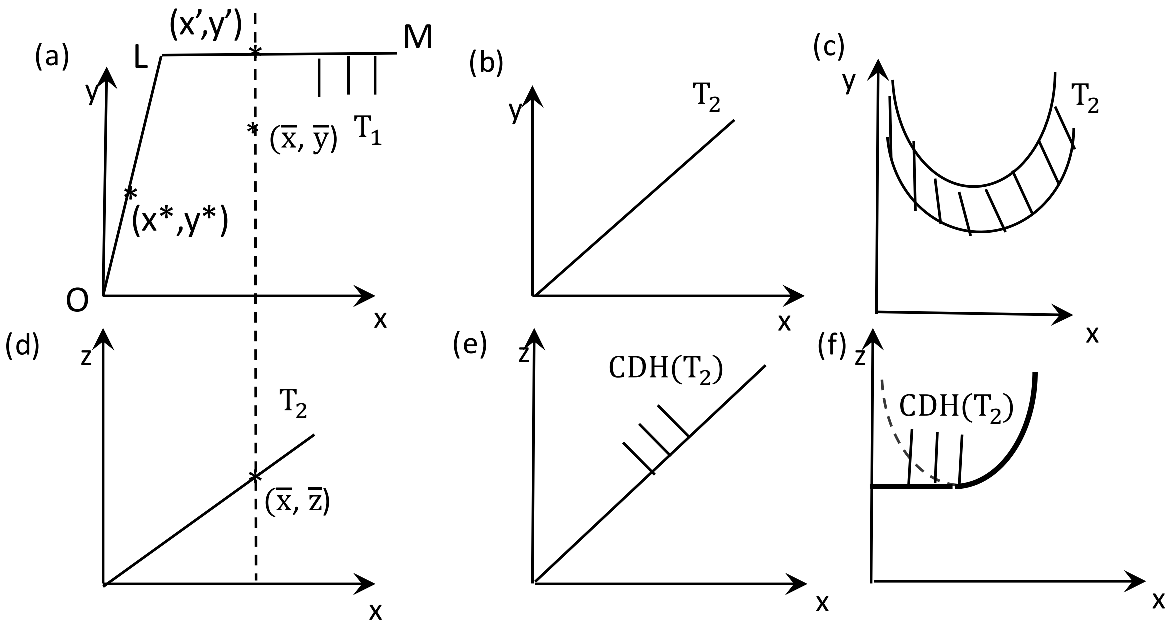


Figure 1

It is clear that the costly disposal hull of any set $A \subset \mathbf{R}^{n+2}_+$ satisfies the assumptions of *costly disposability of emission and emission-causing inputs* as defined in MRL, *i.e.*,[18]

$$\langle x_o, x_z, y, z\rangle \in CDH(A) \ \wedge \ \bar{z} \geq z \ \wedge \ \bar{x}_z \leq x_z \implies \langle x_o, \bar{x}_z, y, \bar{z}\rangle \in CDH(A). \tag{1}$$

Thus, if a production vector belongs to $CDH(A)$, then so does any other production vector with same amounts of the intended output and non-emission causing inputs but with arbitrarily higher amount of the emission and arbitrarily lower amounts of the emission-causing inputs. The remark below follows from Murty and Russell (2017).

**Remark 1** *If a set satisfies costly disposability of emission and emission-causing inputs, then the lower frontier of this set is non-negatively sloped, i.e., the emission level is non-decreasing in emission-causing inputs along the lower frontier of this set. Since the costly disposal hull of any set satisfies costly disposability of emission and emission-causing inputs, its lower frontier is non-negatively sloped.*

Panels (e) and (f) of Figure 1 illustrate that the lower frontier of $CDH(A)$ is non-negatively sloped, where $A = T_2$.

## 2.2 Defining a by-production technology and its efficient frontier.

A by-production technology of MRL, Murty(2015), and Murty and Russell (2016) is an emission-generating technology that is obtained as an intersection of two or more sub-technologies, each

[18]The symbol $\wedge$ stands for "and".

of which captures a distinct rule governing the transformation of inputs into the good and bad outputs. In this paper, which is concerned with the study of coal-based thermal power plants, the by-production technology is defined as an intersection of two sub-technologies. The first, denoted by $T_1 \subset \mathbf{R}^n_+$, is a standard neo-classical technology that describes the transformation of all inputs into the good output, while the second, denoted by $T_2 \subset \mathbf{R}^n_+$, captures considerations such as the laws of thermodynamics, many physical relations, and chemical reactions that govern the generation of the emission due to the use of emission-causing inputs in production.[19] Intended production is based on relations between inputs and intended outputs that are identified by human engineers. Some of the inputs used in intended production are composed of emission-causing substances in proportions determined by nature. When such inputs are employed in intended production, the emission-causing substances are released in various forms depending on their reactions with other substances that come their way. For example, during production of thermal electricity from coal, engineers are concerned with the gross calorific value (GCV) of coal, which measures its heat content, as it is heat energy that is ultimately transformed into thermal electricity in power plants. However, because coal contains carbon, its use for electricity generation also generates $CO_2$ emission. The extent of emission generated depends on the emission-factor of the coal-type employed (which measures the carbon content per unit of coal) and the exposure of the power plant to oxygen.

Here, we provide a stronger definition of a by-production technology as compared to MRL, Murty(2015), and and Murty and Russell (2016).

**Definition 2** *A set $T \subset \mathbf{R}^{N+2}_+$ is a by-production technology (BPT) if there exist two closed sets $T_1 \subset \mathbf{R}^{N+2}_+$ and $T_2 \subset \mathbf{R}^{N+2}_+$ such that the following hold:*

- $T = T_1 \cap T_2$
- *Set $T_1$ satisfies*

  $(i)$ *free disposability of the marketed output and inputs:*

$$\langle x, y, z\rangle \in T_1 \ \wedge \ \bar{y} \leq y \ \wedge \ \bar{x} \geq x \Longrightarrow \langle \bar{x}, \bar{y}, z\rangle \in T_1.$$

  $(ii)$ *independence from emission generation:*

$$\langle x_o, x_z, y, z\rangle \in T_1 \ \wedge \ \bar{z} \neq z \ \Longrightarrow \ \langle x_o, x_z, y, \bar{z}\rangle \in T_1.$$

[19] See also MRL, Murty(2015), and and Murty and Russell (2016,17) for a justification of the by-production approach for specifying emission-generating technologies and for more details about this approach.

*(iii)* *essentiality of all inputs in producing the intended output:*

$$\langle x, y, z\rangle \in T_1 \ \wedge \ x_i = 0 \ \ \textit{for any} \ \ i = 1, \ldots, n \implies y = 0.$$

*(iv)* *convexity.*

- *Set $T_2$ satisfies*

*(i′)* *independence from production of the good output and usage of non-emission causing inputs:*

$$\langle x_o, x_z, y, z\rangle \in T_2 \ \wedge \ \bar{y} \neq y \ \wedge \ \bar{x}_o \neq x_o \implies \langle \bar{x}_o, x_z, \bar{y}, z\rangle \in T_2.$$

*(ii′)* $LFront\,(T_2) = LFront\,(CDH\,(T_2))$.[20]

*(iii′)* *convexity.*

Condition (i) in the definition above is a set of standard disposability assumptions that are imposed on neo-classical technologies.[21] Conditions (ii) and (i') are defined at length in MRL, Murty and Russell (2016, 17). The reader is referred to these works for the motivation behind these assumptions. In particular, condition (ii) implies that sub-technology $T_1$ imposes no restrictions on the level of emission. If a production vector belongs to $T_1$, then so does any other production vector with the same amounts of inputs and the intended outputs but with *any other* amount of the emission. Thus, it is assumed that the production of the intended output is unaffected by the level of emission.[22]

The assumption of essentiality in condition (iii) of the above definition of a BPT implies that all inputs are important for producing the intended output, in the sense that, if any one of the input is not used then the output cannot be produced.

Condition (i') says that sub-technology $T_2$ imposes no restrictions on the levels of the non-emission causing inputs and the intended output. Thus, emission generation is not directly caused by the intended output or non-emission causing inputs.

Condition (ii') has been employed in the definition of a BPT to focus on most real-life cases where emission generation increases with increase in use of some specific inputs, *e.g.* $CO_2$ emission level is increasing in the usage of coal. This condition, which has been discussed at length in Murty and Nagpal (2018), requires the lower frontier of sub-technology $T_2$ to coincide

[20] In other words the minimum feasible levels of emission under $T_2$ and its costly disposal hull are the same for all vectors of emission-causing inputs, *i.e.*, $G\,(x_z;\ T_2) = G\,(x_z;\ CDH\,(T_2))$

[21] See, for instance, Shephard (1953), Debreu (1959), and Mas-Colell et al. (1995) Chapter 5.

[22] For a treatment of a more general case under the by-production approach, where it can be, see Murty (2015).

with the lower frontier of its costly disposal hull.[23] In Figure 1, panel-pair (b) and (e) is an example of a situation where condition (ii') holds. Since the lower frontiers of sub-technology $T_2$ and its costly disposal hull coincide in this case, Remark 1 implies that the lower frontier of sub-technology $T_2$ is also non-negatively sloped. In panel-pair (c) and (f), condition (ii') fails to hold and the lower frontier of sub-technology $T_2$ has a downward sloping region. This is indicated by the dashed curve in panel (f), which is not a part of the lower frontier of the costly disposal hull of sub-technology $T_2$.

The following remark, which follows as a straightforward implication of Remark 1 and conditions (i') and (ii') in Definition 2 of a BPT, summarises the above point.

**Remark 3** *If set $T_2 \subset \mathbf{R}_+^{n+2}$ satisfies (i') and (ii') in Definition 2 then the lower frontier of set $T_2$ is non-negatively sloped, i.e., the emission level is non-decreasing in emission-causing inputs along $LFront\,(T_2)$. Equivalently, $G\,(x_z; T_2)$ is non-decreasing function of $x_z$.*

In the rest of the paper, $T \subset \mathbf{R}_+^{n+2}$ will denote a BPT and sub-technologies $T_1 \subset \mathbf{R}_+^{n+2}$ and $T_2 \subset \mathbf{R}_+^{n+2}$ will satisfy all conditions in Definition 2.

We say that a production vector $\langle x_o, x_z, y, z\rangle$ in set $T$ (respectively, sub-technology $T_1$) is an *efficient point* of $T$ (respectively, sub-technology $T_1$) if there exists no other point in $T$ (respectively, sub-technology $T_1$) with no bigger amounts of the inputs and emission and no smaller amount of the good output.[24]

A production vector $\langle x_o, x_z, y, z\rangle \in T$ (respectively, sub-technology $T_1$) is a *weakly efficient point* of $T$ (respectively, sub-technology $T_1$) if there exists no other point in $T$ (respectively, sub-technology $T_1$) with strictly smaller amounts of the inputs and emission and bigger amount of the good output.[25]

It is clear that if a production vector $\langle x_o, x_z, y, z\rangle \in T$ is an efficient point of $T$ then it is a weakly efficient point of $T$ (respectively, sub-technology $T_1$). The converse, however, may not be true. In panel (a) of Figure 1, $\langle x', y'\rangle$ is a weakly efficient but not an efficient point of sub-technology $T_1$, while $\langle x', y'\rangle$ is both an efficient and a weakly efficient point in it. On the other hand, $\langle \bar{x}, \bar{y}\rangle$ is an inefficient point of sub-technology $T_1$.

The set of all efficient points of $T$ (respectively, $T_1$) form the *efficient frontier* of $T$ (respectively, $T_1$). The set of all weakly efficient points of $T$ (respectively, $T_1$) form the *weakly efficient frontier of* $T$ (respectively, $T_1$).

---

[23]See Murty and Nagpal (2018) for further details and illustrations.

[24]That is, if there exists no other point $\langle x'_o, x'_z, y', z'\rangle$ in $T$ (respectively, sub-technology $T_1$) such that $\langle x'_o, x'_z\rangle \leq \langle x_o, x_z\rangle$, $y' \geq y$, and $z' \leq z$.

[25]That is, if there exists no other point $\langle x'_o, x'_z, y', z'\rangle$ in $T$ (respectively, sub-technology $T_1$) such that $\langle x'_o, x'_z\rangle \ll \langle x_o, x_z\rangle$, $y' > y$, and $z' < z$.

# 3 A class of weighted indexes of efficiency improvements.

In this section we define a class of weighted indexes of inefficiency, which can also be interpreted as weighted indexes of efficiency improvements. To do so, we first define the set of all non-negative weights on inputs and outputs which sum to one as the unit simplex $\Delta \in \mathbf{R}_+^{n+2}$

$$\Delta = \Big\{ w = \langle w_o^x, w_z^x, w^y, w^z \rangle \in \mathbf{R}_+^{n+2} \;\Big|\; \sum_{i=1}^{n_o} w_{o_i}^x + \sum_{i=1}^{n_z} w_{z_i}^x + w^y + w^z = 1 \Big\},$$

where $w_o^x \in \mathbf{R}_+^{n_o}$, $w_z^x \in \mathbf{R}_+^{n_z}$, $w^y$, and $w^z$ denote the weights on non-emission and emission-causing inputs and the good and bad outputs, respectively.

## 3.1 A weighted index of graph efficiency improvements.

As in MRL, let $\otimes$ denote an operator in any $m$-dimensional Euclidean space $\mathbf{R}_+^m$ such that, for any two vectors $a$ and $b$ in $\mathbf{R}_+^m$, we have

$$a \otimes b \;=\; \langle a_1 b_1, \;\ldots\;, \; a_m b_m \rangle.$$

Define the weighted index of graph inefficiency/efficiency-improvements as a mapping $\mathcal{I}^G$ : $T \times \Delta \longrightarrow \mathbf{R}_+$ with image

$$\begin{aligned} \mathcal{I}^G(x_o, x_z, y, z; w) = \max_{\langle \delta_o, \delta_z, \theta, \gamma \rangle} \; \sum_{i=1}^{n_o} w_{o_i}^x \delta_{o_i} + \sum_{i=1}^{n_z} w_{z_i}^x \delta_{z_i} + w^y \theta + w^z \gamma =: W\left(\delta_o, \delta_z, \theta, \gamma; w\right) \\ \text{subject to} \\ \big\langle x_o - (\delta_o \otimes x_o)\,, \quad x_z - (\delta_z \otimes x_z)\,, \quad y + \theta y, \quad z - \gamma z \big\rangle \in T = T_1 \cap T_2, \qquad (2) \\ \delta_o \in [0,1]^{n_o}, \quad \delta_z \in [0,1]^{n_z}, \quad \gamma \in [0,1], \quad \theta \geq 0, \end{aligned}$$

where $W$ denotes the objective function of problem (2).[26] It is a weighted sum of proportional changes in all inputs and outputs, where the vectors of proportional changes in the non-emission causing and emission-causing inputs are respectively given by $\delta_o \in \mathbf{R}^{n_o}$ and $\delta_z \in \mathbf{R}^{n_z}$, while the proportional changes in the intended and bad outputs are respectively denoted by $\theta$ and $\gamma$, both in $\mathbf{R}$. Thus, $W$ is a function of the vector of proportional changes in inputs and outputs and the vector of weights in $\Delta$. Starting from any given production vector $v := \langle x_o, x_z, y, z \rangle \in T$, problem (2) finds the vector of proportional changes in inputs and outputs that are (i) feasible under the BPT $T$, (ii) result in no-bigger amounts of inputs and emission and no lower amount

[26]That is, $W\left(\delta_o, \delta_z, \theta, \gamma; w\right) = \sum_{i=1}^{n_o} w_{o_i}^x \delta_{o_i} + \sum_{i=1}^{n_z} w_{z_i}^x \delta_{z_i} + w^y \theta + w^z \gamma$.

of the intended output, and (iii) maximise the weighted sum $W$. The implication of (i) and (ii) is that problem (2) restricts $\delta_i$ to lie in the closed interval $[0,1]$ for all $i = 1, \ldots, n$. $x_i \geq x_i - \delta_i x_i$ for all $i = 1, \ldots, n$. The upper bound of one ensures that the new level of input $i$ is non-negative. Exactly the same restrictions also apply on the possible values that $\gamma$ (the proportional change in emission) can take in problem (2). To ensure that the new level of the intended output is no smaller than the original level $y$, this problem restricts $\theta$ to take non-negative values, as this would imply that $y \leq y + \theta y$.

It is clear that if a vector $s := \langle \delta_o, \delta_z, \theta, \gamma \rangle$ of proportional changes in inputs and good and bad outputs is a solution to problem (2), then $v_s := \left\langle x_o - (\delta_o \otimes x_o)\,,\;\; x_z - (\delta_z \otimes x_z)\,,\;\; y + \theta y,\;\; z - \gamma z \right\rangle$ is an efficient point of $T$. Intuitively, the greater is the value of the weighted inefficiency index $\mathcal{I}^G(x_o, x_z, y, z; w)$, the higher is the maximum possible value that the weighted sum $W$ of feasible proportional changes in inputs and outputs can take, and hence, the further is the original production vector $v$ away from the efficient frontier of technology $T$.[27]

At the same time, the solution vector $s$ can also be interpreted as a vector of efficiency improvements for the producing unit. Starting from its original production vector $v \in T$, it gives the optimal proportional reductions in inputs and emission levels and the optimal proportional increase in the good output given the objective function $W$ and the chosen vector of weights $w \in \Delta$. If the production unit changes its production vector in the direction of the solution vector of proportional changes $s$, then its technical efficiency increases. Hence, we also call the weighted sum of these proportional changes $\mathcal{I}^G$ as the weighted index of graph efficiency improvements.

### 3.2 An output-based index of weighted efficiency improvements.

Output-based measures of (in)efficiency are derived holding the levels of all inputs fixed. For comparability between the weighted index of graph efficiency improvements and the output-based efficiency indexes derived in MRL, we also define a weighted index of output-based efficiency improvements as a function: $\mathcal{I}^O : T \times \Delta \longrightarrow [1, \infty)$ with image

[27] In the context of the by-production approach, Dapko (2015) brings to light some issues pertaining to defining efficiency improvements in the input directions (see Murty and Russell (2017) for a summary). Noting these issues, Murty and Russell (2017) propose an extension of the output-based FGL *efficiency* index to the graph space (full space of inputs and good and bad outputs). The index $\mathcal{I}^G$ can be interpreted to be its dual weighted graph *inefficiency* index.

$$
\begin{aligned}
\mathcal{I}^O(x_o, x_z, y, z; w) \;&=\; \max_{\langle \delta_o, \delta_z, \theta, \gamma \rangle} W(\delta_o, \delta_z, \theta, \gamma; w) \\
&\quad \text{subject to} \\
&\quad \Big\langle x_o - (\delta_o \otimes x_o), \quad x_z - (\delta_z \otimes x_z), \quad y + \theta y, \quad z - \gamma z \Big\rangle \in T = T_1 \cap T_2, \\
&\quad \delta_o = 0_{n_o}, \quad \delta_z = 0_{n_z}, \quad \gamma \in [0,1], \quad \theta \geq 0, \qquad (3)
\end{aligned}
$$

Thus, problem (3) is constructed from problem (2) by adding an extra restriction that all inputs are held fixed, *i.e.*, no changes in the input-levels are permitted: $\delta_o = 0_{n_o}$ and $\delta_z = 0_{n_z}$.

It can be shown that the output-based inefficiency index $\mathcal{I}^O$ is equivalent to

$$
\begin{aligned}
\mathcal{I}^O(x_o, x_z, y, z; w) \;&=\; w^y\Big[\boldsymbol{\beta}_y(x_o, x_z, y, z) - 1\Big] + w^z\Big[1 - \boldsymbol{\beta}_z(x_o, x_z, y, z)\Big] \\
&\equiv\; w^y[\beta_y - 1] + w^z[1 - \beta_z], \qquad (4)
\end{aligned}
$$

where functions $\boldsymbol{\beta}_y : T_1 \longrightarrow [1, \infty)$ and $\boldsymbol{\beta}_z : T_2 \longrightarrow [0,1]$ are, respectively, the output-based productive inefficiency index and the environmental efficiency index that are defined in MRL:

$$
\beta_y = \boldsymbol{\beta}_y(x_o, x_z, y, z) = \sup\Big\{\beta \geq 0 \;\Big|\; \langle x_o, x_z, \beta y, z\rangle \in T_1\Big\} \quad \text{and} \qquad (5)
$$

$$
\beta_z = \boldsymbol{\beta}_z(x_o, x_z, y, z) = \inf\Big\{\beta \geq 0 \;\Big|\; \langle x_o, x_z, y, \beta z\rangle \in T_2\Big\}. \qquad (6)
$$

Function $\boldsymbol{\beta}_y$ gives the maximum amount by which the good output can be expanded under sub-technology $T_1$, when inputs are held fixed, while the inverse of function $\boldsymbol{\beta}_z$ gives the maximum amount by which the bad output can be contracted under sub-technology $T_2$, when inputs are held fixed. Thus, $\beta_y \geq 1$, while $\beta_z \in [0,1]$, and the production vector $v_\beta := \langle x_o, x_z, \beta_y y, \beta_z z\rangle$ lies on the weakly efficient frontier of $T_1$ and the lower frontier of $T_2$.[28]

The proportional increase in the good output and the proportional reduction in the bad output in moving from the original production vector $v$ to the the production vector $v_\beta$ are hence given by $\beta_y - 1$ and $1 - \beta_z$, respectively. Thus, (4) implies that the output-based weighted index $\mathcal{I}^O$ can be interpreted as the weighted sum of both these proportional changes (efficiency-improvements) in the good and bad-output directions.

It is helpful to define also the output-based measure of *productive efficiency* with respect to

[28] This follows from the disposability properties of sub-technologies $T_1$ and $T_2$ specified in Definition 2.

sub-technology $T_1$ as the inverse of output-based index of productive inefficiency $\boldsymbol{\beta}_y$:

$$b_y = \mathbf{b}_y\left(x_o, x_z, y, z\right) = \frac{1}{\boldsymbol{\beta}_y(x_o, x_z, y, z)} = \frac{1}{\beta_y}. \tag{7}$$

Since function $\boldsymbol{\beta}_y$ takes values that are no smaller than one, function $\mathbf{b}_y$ takes values lying between zero and one. The more closer its value to one, the more efficient (less inefficient) is the producing unit with respect to sub-technology $T_1$.

The remark below follows from the definitions of weighted graph index $\mathcal{I}^G$ and output-based indexes $\boldsymbol{\beta}_y$ and $\boldsymbol{\beta}_z$. It says that, when a production vector is an efficient point of technology $T$, then there exist no feasible efficiency improvements in inputs and outputs, so that the solution to problem (2) is a zero vector and the inefficiency index $\mathcal{I}^G$ takes a value zero. Further, when a production vector lies on the efficient frontier of $T_1$, then it is not possible to reduce usage of inputs and increase the production of good output under this sub-technology. Lastly, when output-based index $\boldsymbol{\beta}_y$ (respectively, $\boldsymbol{\beta}_z$) takes a value equal to one, then the production vector lies on the weakly efficient frontier (respectively, lower frontier) of sub-technology $T_1$ (respectively, sub-technology $T_2$).[29]

**Remark 4** *Consider a production vector* $\bar{v} = \langle \bar{x}_o, \bar{x}_z, \bar{y}, \bar{z} \rangle \in T$ *and let* $\bar{s} = \langle \bar{\delta}_o, \bar{\delta}_z, \bar{\theta}, \bar{\gamma} \rangle$ *be a solution to problem (2) for* $\langle x_o, x_z, y, z \rangle = \bar{v}$*. Let* $\bar{\beta}_y = \boldsymbol{\beta}_y\left(\bar{x}_o, \bar{x}_z, \bar{y}, \bar{z}\right)$ *and* $\bar{\beta}_z = \boldsymbol{\beta}_z\left(\bar{x}_o, \bar{x}_z, \bar{y}, \bar{z}\right)$*. Then* $\bar{v}$ *is*

- *an efficient point of technology* $T$ *if and only if* $\bar{s} = 0_{n+2}$*. In that case, we have,* $\mathcal{I}^G(\bar{x}_o, \bar{x}_z, \bar{y}, \bar{z}; w) = 0$*.*
- *a weakly efficient point of sub-technology* $T_1$ *if and only if* $\bar{\beta}_y = 1$*. It is an efficient point of sub-technology* $T_1$ *if and only if* $\langle \bar{\delta}_o, \bar{\delta}_z, \bar{\theta} \rangle = 0_{n+1}$*.*
- *a point on the lower frontier of sub-technology* $T_2$ *if and only if* $\bar{\beta}_z = 1$*.*

# 4 Maximum proportional increase in good output and decrease in emission given proportional reductions in inputs.

In this section we derive the maximum feasible proportional increase in the intended output and the maximum feasible proportional reduction in the emission level for a given vector of

[29] For any positive integer $k$, $0_k$ denotes a $k$-dimensional vector of zeroes.

proportional reductions in inputs. As will be seen in Sections 5, 6, and 9, these concepts are useful in the qualitative characterisation of all possible solutions to problem (2).

Denote the sets $[0,1]^{n_z}$ and $[0,1]^n$ by $\mathfrak{Q}_z$ and $\mathfrak{Q}$, respectively.[30] Fix a point in the BPT, say $\bar{v} = \langle \bar{x}_o, \bar{x}_z, \bar{y}, \bar{z} \rangle \in T$ and define functions $\Theta : \mathfrak{Q} \longrightarrow \mathbf{R}$ and $\Gamma : \mathfrak{Q}_{n_z} \longrightarrow \mathbf{R}$ with images:

$$\Theta(\delta_o, \delta_z) := \max_{\theta \in \mathbf{R}} \{\theta \mid \left\langle \bar{x}_o - (\delta_o \otimes \bar{x}_o),\ \bar{x}_z - (\delta_z \otimes \bar{x}_z),\ \bar{y} + \theta\bar{y},\ \bar{z} \right\rangle \in T_1\} \tag{8}$$

$$\Gamma(\delta_z) := \max_{\gamma \in \mathbf{R}} \{\gamma \mid \left\langle \bar{x}_o,\ \bar{x}_z - (\delta_z \otimes \bar{x}_z),\ \bar{y},\ \bar{z} - \gamma\bar{z} \right\rangle \in T_2\}. \tag{9}$$

Starting from $\bar{x}$, function $\Theta$ gives the maximum proportional increase in the economic output that is feasible under sub-technology $T_1$ for every given vector of proportional decreases in inputs $\delta$ lying in set $\mathfrak{Q}$. Similarly, function $\Gamma$ gives the maximum proportional decrease in the level of emission that is feasible under sub-technology $T_2$ for every given vector of proportional decreases in emission-causing inputs $\delta_z \in \mathfrak{Q}_z$ in inputs.

Figures 2 and 3 explain the intuition behind functions $\Theta$ and $\Gamma$. Figure 2 assumes that there is a single input that is emission-causing, *i.e.,* $n = n_z = 1$, while Figure 3 assumes that there are two inputs, both of which are emission-causing, *i.e.,* $n = n_z = 2$. Starting from $\bar{x}$, if the input vector is reduced by a proportion equal to $\hat{\delta} \in \mathfrak{Q}$, then the new vector of inputs is $\hat{x} = \bar{x} - \hat{\delta}\bar{x}$, which is lower than $\bar{x}$. Panel (a) of Figure 2 and Figure 3 show that, when the input vector is $\hat{x}$ then, starting from $\bar{y}$ level of the intended output, the maximum possible increase in the intended output under sub-technology $T_1$ is given by $\hat{\theta}\bar{y}$. Thus, the maximum proportional increase in the intended output due to a proportional decrease $\hat{\delta}$ in the inputs is $\Theta\left(\hat{\delta}\right) = \hat{\theta}$, and the new intended output level is $\bar{y} + \hat{\theta}\bar{y}$. Similarly, panel (b) of Figure 2 shows that, starting from $\bar{z}$ level of emission, the maximum reduction in the emission under sub-technology $T_2$ is given by $\hat{\gamma}\bar{z}$ when the input level is $\hat{x}$. Thus, the maximum proportional decrease in emission due to a proportional decrease $\hat{\delta}$ in the inputs is $\Gamma\left(\hat{\delta}\right) = \hat{\gamma}$, and the new level of emission is $\bar{z} - \hat{\gamma}\bar{z}$, which is lower than $\bar{z}$.

The following theorem states the monotonicity properties of functions $\Theta$ and $\Gamma$, which are discussed in detail in the next two subsections.

**Theorem 5** *$\Theta$ is a non-increasing function and $\Gamma$ is a non-decreasing function. Both functions are concave.*

[30]Note that, if a vector of proportional changes in inputs $\delta \in \mathbf{R}^n_+$ lies in $\mathfrak{Q} = [0,1]^n$, then every element of $\delta$ lies in the interval $[0,1]$. Similarly, if a vector of proportional changes in emission-causing inputs $\delta_z \in \mathbf{R}^{n_z}_+$ lies in $\mathfrak{Q}_z = [0,1]^{n_z}$, then every element of $\delta_z$ lies in the interval $[0,1]$.

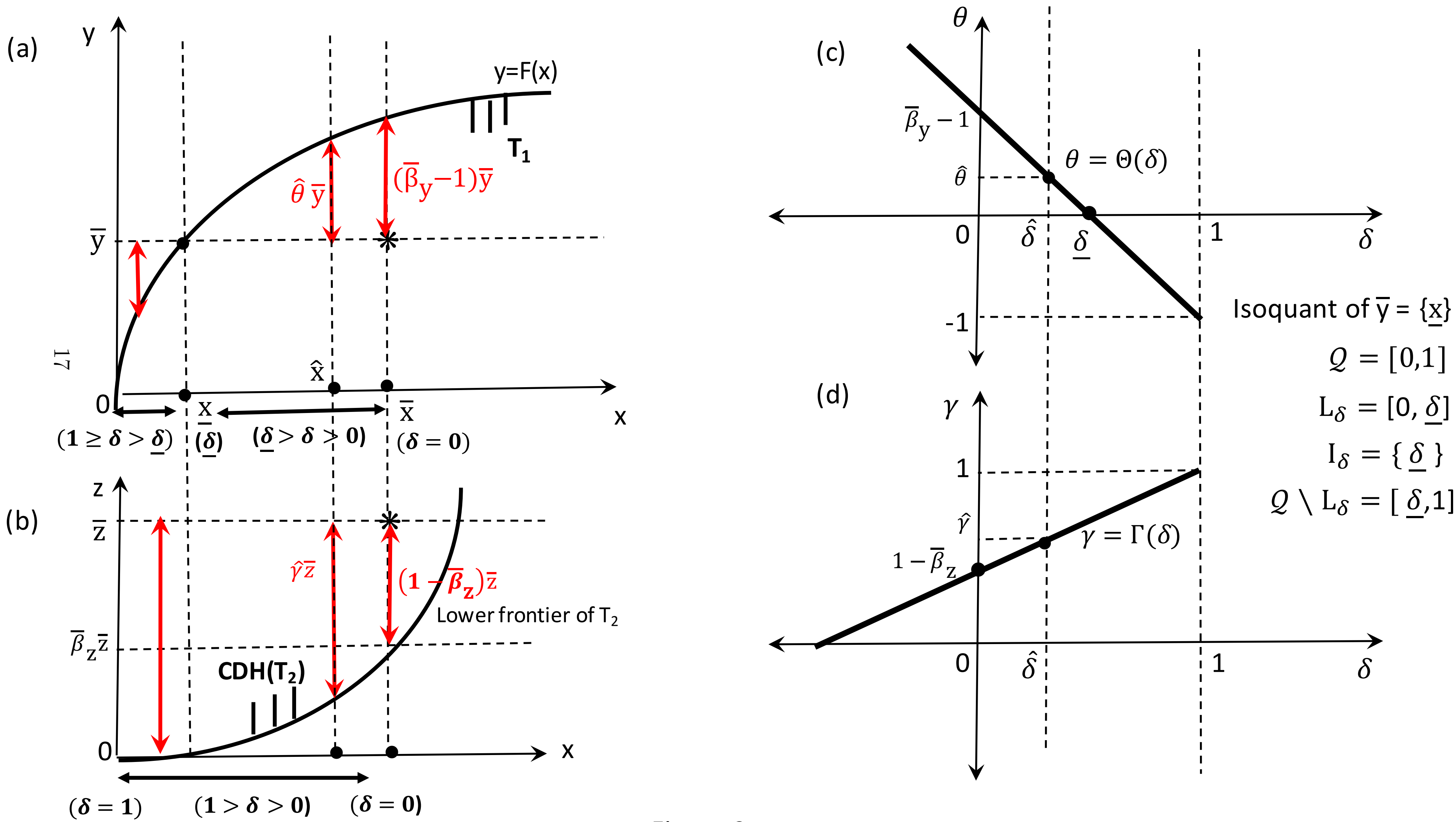
(a)
y
x
0
y=F(x)
$T_1$
$\bar{y}$
$\hat{\theta}\,\bar{y}$
$(\bar{\beta}_y - 1)\bar{y}$
$\underline{x}$
$\hat{x}$
$\bar{x}$
$(1 \geq \delta > \underline{\delta})$
$(\underline{\delta})$
$(\underline{\delta} > \delta > 0)$
$(\delta = 0)$
(b)
z
$\bar{z}$
$\bar{\beta}_z \bar{z}$
0
x
$\hat{\gamma}\bar{z}$
$(1 - \bar{\beta}_z)\bar{z}$
Lower frontier of $T_2$
$CDH(T_2)$
$(\delta = 1)$
$(1 > \delta > 0)$
$(\delta = 0)$
(c)
$\theta$
$\bar{\beta}_y - 1$
$\hat{\theta}$
0
-1
$\hat{\delta}$
$\underline{\delta}$
1
$\delta$
$\theta = \Theta(\delta)$
Isoquant of $\bar{y}$ = {$\underline{x}$}
$\mathcal{Q} = [0,1]$
$L_\delta = [0, \underline{\delta}]$
$I_\delta = \{ \underline{\delta} \}$
$\mathcal{Q} \setminus L_\delta = [\underline{\delta}, 1]$
(d)
$\gamma$
1
$\hat{\gamma}$
$1 - \bar{\beta}_z$
$\gamma = \Gamma(\delta)$
0
$\hat{\delta}$
1
$\delta$

Figure 2

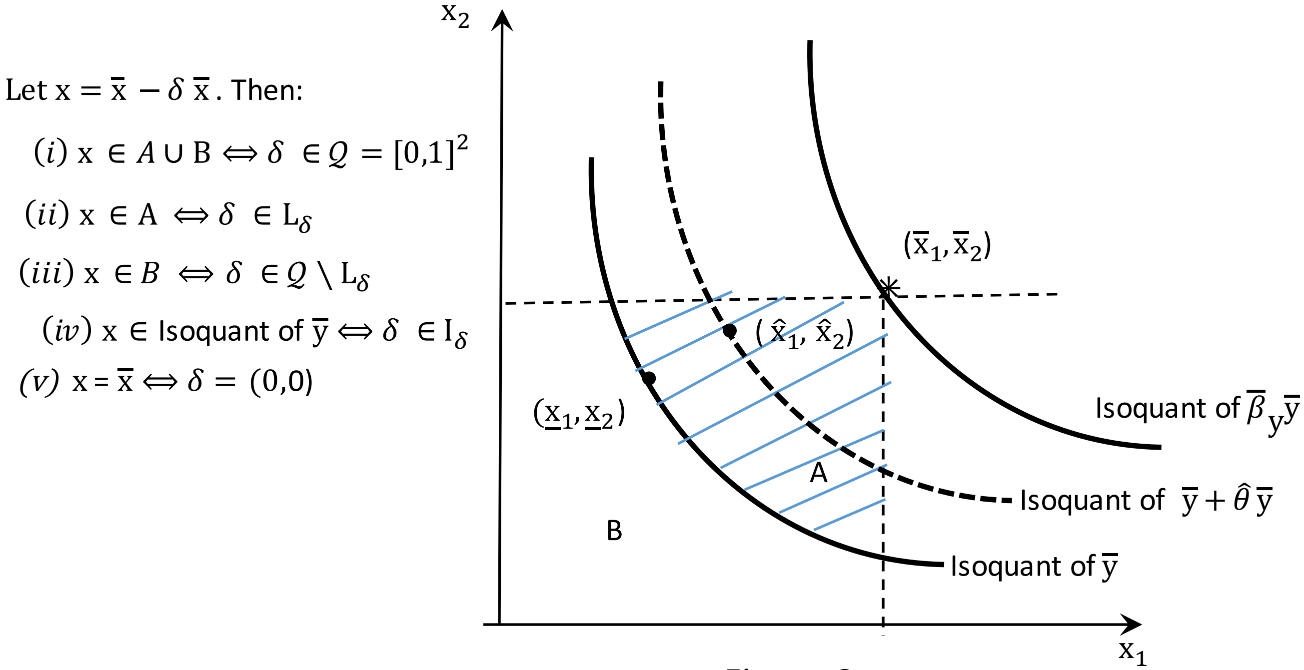


Figure 3

### 4.1 Image of function $\Theta$.

To study the image of function $\Theta$, we will first partition its domain $\mathcal{Q}$ into four mutually exclusive sets. To this end, we first derive the following production function from sub-technology $T_1$ by maximising the production of intended output under sub-technology $T_1$ given any vector of input levels.[31]

$$F(x_o, x_z) := \max\{y \geq 0 \mid \langle x_o, x_z, y, z\rangle \in T_1\}.$$

Under the free disposability condition (i) in Definition 2, function $F$ can be shown to be non-decreasing in all inputs and it can be employed to functionally represent sub-technology $T_1$.[32] The input requirement set of sub-technology $T_1$ corresponding to $\bar{y}$ level of the intended output can be written as

$$L\left(\bar{y}\right) = \Big\{\langle x_o, x_z\rangle \in \mathbf{R}^n_+ \;\Big|\; \bar{y} \leq F(x_o, x_z)\Big\}.$$

It is the set of all input bundles that can produce $\bar{y}$ level of the intended output under sub-technology $T_1$. We now define the set

$$L_\delta \;\; = \;\; \Big\{\langle \delta_o, \delta_z\rangle \in \mathcal{Q} \;\Big|\; \bar{y} \leq F\left(\bar{x}_o - (\delta_o \otimes \bar{x}_o)\,,\; \bar{x}_z - (\delta_z \otimes \bar{x}_z)\right)\Big\} \qquad (10)$$

[31]Since $T_1$ satisfies independence from the emission level, $z$ is not shown as an argument of function $F$.
[32]See, for instance, Russell (1998) and Murty and Russell (2017).

as the set of proportional changes in inputs lying in $\mathfrak{Q}$ such that the corresponding changed levels of input can continue producing intended output level $\bar{y}$ under sub-technology $T_1$. That is, for any $\langle \delta_o, \delta_z \rangle \in L_\delta$, the input vector $\big\langle \bar{x}_o - (\delta_o \otimes \bar{x}_o)\,,\ \bar{x}_z - (\delta_z \otimes \bar{x}_z)\,\big\rangle$ lies in the input requirement set $L\,(\bar{y})$. Since $\bar{v} \in T$, we have $\bar{y} \leq F\,(\bar{x}_o, \bar{x}_z)$. Hence, it is clear that the vector of no proportional changes in inputs (*i.e.*, $0_n$) is in $L_\delta$. The isoquant of sub-technology $T_1$ corresponding to $\bar{y}$ level of intended output is

$$I\,(\bar{y}) = \big\{\langle x_o, x_z\rangle \in \mathbf{R}^n_+ \mid F(x_o, x_z) = \bar{y}\big\}$$

It is the set of all input combinations such that the maximum intended output produced by each of these combinations under sub-technology $T_1$ is $\bar{y}$. The following subset of $L_\delta$ is the set of all proportional changes in inputs in $\mathfrak{Q}$ that result in points on the isoquant of $\bar{y}$ starting from $\langle \bar{x}_o, \bar{x}_z\rangle$:

$$I_\delta = \left\{\langle \delta_o, \delta_z\rangle \in \mathfrak{Q} \;\middle|\; F\,(\bar{x}_o - (\delta_o \otimes \bar{x}_o)\,,\ \bar{x}_z - (\delta_z \otimes \bar{x}_z)) = \bar{y}\right\}.$$

Note that the set $I_\delta$ may or may not include $0_n$. If $I_\delta$ includes $0_n$ then $F\,(\bar{x}_o, \bar{x}_z) = \bar{y}$ and production vector $\bar{v} = \langle \bar{x}_o, \bar{x}_z, \bar{y}, \bar{z}\rangle$ is a weakly efficient point of sub-technology $T_1$.

It follows from the above discussion that $L_\delta$ can be decomposed into (i) set $I_\delta$ after excluding the zero vector, (ii) the singleton set containing the zero vector, and (iii) the part remaining of set $L_\delta$ after excluding (i) and (ii)[33]

$$L_\delta \;=\; I_\delta \setminus \{0_n\} \;\cup\; \{0_n\} \;\cup\; L_\delta \setminus (I_\delta \cup \{0_n\})\,. \tag{11}$$

From this it follows that set $\mathfrak{Q}$ can be partitioned into four mutually exclusive and exhaustive subsets, namely the subset of $\mathfrak{Q}$ that excludes all elements in $L_\delta$, (which can be denoted by $\mathfrak{Q} \setminus L_\delta$) and the three mutually exclusive and exhaustive components of set $L_\delta$ in (11). The values that function $\Theta$ takes in these four subsets of its domain are given by:

$$\begin{aligned}
\Theta\,(\delta_o, \delta_z) \;&\in\; [-1, 0) && \forall\ \langle \delta_o, \delta_z\rangle \in \mathfrak{Q} \setminus L_\delta \\
&=\; 0 && \forall\ \langle \delta_o, \delta_z\rangle \in I_\delta \setminus \{0_n\} \\
&\in\; \big(0, \bar{\beta}_y - 1\big] && \forall\ \langle \delta_o, \delta_z\rangle \in L_\delta \setminus (I_\delta \cup \{0_n\}) \\
&=\; \bar{\beta}_y - 1 && \text{for } \langle \delta_o, \delta_z\rangle = 0_n.
\end{aligned} \tag{12}$$

[33]Given sets $A$ and $B$, the notation $A \setminus B$ stands for a set containing all elements of set $A$ excluding those in set $B$.

To understand the image of function $\Theta$ presented above, first recall that $\bar{\beta}_y$ denotes the value of the output-based inefficiency index $\boldsymbol{\beta}_y(\bar{x}_o, \bar{x}_z, \bar{y}, \bar{z})$. If $\delta = \langle \delta_o, \delta_z \rangle = 0_n$, then there is no change in the input levels and the maximum proportional increase in output (*i.e.,* the value that $\Theta$ can take) is given by $\bar{\beta}_y - 1$. If $\delta \in I_\delta$, *i.e.,* if the new levels of inputs lie in the isoquant of $\bar{y}$, then the maximum output they can produce is $\bar{y}$ itself so that $\Theta$ takes a value zero. On the other hand, if $\delta \in L_\delta \setminus (I_\delta \cup \{0_n\})$, then the new levels of inputs lie in the input requirement set of $\bar{y}$ but are not in its isoquant. This means that they can produce intended output levels higher than $\bar{y}$, so that $\Theta$ takes values greater than zero. When $\delta \in \Omega \setminus L_\delta$, *i.e.,* the new input vector does not lie in the input requirement set of $\bar{y}$, then $\Theta$ takes negative values. This is because, in this case, the proportional reductions in the inputs given by $\delta$ are so high that the new levels of inputs can no longer produce the intended output level $\bar{y}$.

Figures 2 and 3 illustrate these cases. In both the figures, sets $\Omega$, $L_\delta$ and $I_\delta$ are indicated. The isoquant of $\bar{y}$ is the singleton set $\{\underline{x}\}$ in panel (a) of Figure 2. This implies that set $I_\delta$ also contains only one element, which we denote by $\underline{\delta}$, that solves $\underline{x} = \bar{x} - \underline{\delta}\bar{x}$. In Figure 3, the isoquant of $\bar{y}$, and hence the set $I_\delta$, have many points. It is to be noted that in both Figures 2 and 3, $I_\delta$ does not include zero, *i.e.,* $\bar{x}$ does not lie on the isoquant of $\bar{y}$. But $\bar{x}$ does belong to the the input requirement set of $\bar{y}$, $L(\bar{y})$, so that zero is an element of $L_\delta$. When $\delta = 0_n$ (where $n = 1$ in Figure 2 and $n = 2$ in Figure 3), there is no change in the input level(s), and the maximum increase in the intended output starting from its initial level $\bar{y}$ is $\left(\bar{\beta}_y - 1\right)\bar{y}$. As seen clearly in panel (a) of Figure 2, starting from $\delta = 0$, increases in $\delta$ reduce the input level below $\bar{x}$ and reduce the maximum proportionate increase in the intended output below $\bar{\beta}_y - 1$. When $\delta = \underline{\delta}$, the new input level is $\underline{x}$, which lies in the isoquant of $\bar{y}$ and so the maximum proportionate increase in the intended output when input level is $\underline{x}$, is zero. For further increase in $\delta$ beyond $\underline{\delta}$, the maximum increase in the intended output is negative, *i.e.,* the output falls below $\bar{y}$. Panel (c) of Figure 2 plots function $\Theta$. In this example, the graph of $\Theta$ is negatively sloped, which is consistent with Theorem 5.

## 4.2 Image of function $\Gamma$.

Non-negativity of technologically feasible levels of the emission implies that $\Gamma(\delta_z)$ has to take values that are no-bigger than one. The image of function $\Gamma$ is as follows:

$$\begin{aligned} \Gamma(\delta_z) &\in [1 - \bar{\beta}_z, 1] \quad \forall\ \delta_z \in \Omega_z \setminus \{0_{n_z}\} \\ &= 1 - \bar{\beta}_z \qquad \text{if } \delta_z = 0_{n_z} \end{aligned} \tag{13}$$

Function $\Gamma$ is illustrated in panels (b) and (d) of Figure 2, which assumes that the single input is emission-causing. When $\delta = 0$ (which is equivalent to no change in the input level from

$\bar{x}$), the maximum possible reduction in emission level starting from $\bar{z}$ is given by $(1-\bar{\beta}_z)\bar{z}$, where $\bar{\beta}_z$ is the value of the output-based efficiency index $\boldsymbol{\beta}_z$ defined in (6) evaluated at $\bar{v}$, *i.e.,* $\boldsymbol{\beta}_z(\bar{x}_o, \bar{x}_z, \bar{y}, \bar{z}) = \bar{\beta}_z$. Hence, $\Gamma(0) = 1-\bar{\beta}_z$. Panels (b) and (d) in Figure 2 show that, as $\delta$ increases starting from zero, the input level falls below $\bar{x}$, and the maximum proportionate reduction in emission $\Gamma(\delta)$ increases starting from $1-\bar{\beta}_z$. This is because, in this example, the lower frontier of sub-technology $T_2$ is negatively sloped. This is consistent with Remark 3 that holds under our maintained assumptions.

# 5 First-order conditions of problem (2) and their interpretations.

The following proposition relates problem (2) that computes the graph index of weighted inefficiency/efficiency improvements $\mathcal{I}^G$ to problems (8) and (9) that compute functions $\Theta$ and $\Gamma$, respectively.

**Proposition 6** *Suppose* $\bar{s} = \langle \bar{\delta}_o, \bar{\delta}_z, \bar{\theta}, \bar{\gamma}\rangle$ *solves problem (2) given production vector* $\bar{v} = \langle \bar{x}_o, \bar{x}_z, \bar{y}, \bar{z}\rangle \gg 0^{n+2}$. *Then we have (a)* $\langle \bar{\delta}_o, \bar{\delta}_z\rangle \in [0,1)^n$, *(b)* $\bar{\theta} = \Theta(\bar{\delta}_o, \bar{\delta}_z) \geq 0$, *and (c)* $\bar{\gamma} = \Gamma(\bar{\delta}_z)$.

First note that part (a) in the above proposition implies that, at a solution $\bar{s}$ to problem (2), $\bar{\delta}_i$ is *strictly* less than one for all $i = 1, \ldots, n$. From this it follows that each element of the implied new input vector, given by $\bar{x} - (\bar{\delta} \otimes \bar{x})$, is *strictly* bigger than zero. This follows from condition (iii) in Definition 2 of a BPT. This is because, if all inputs are essential, then reducing any input $i = 1, \ldots, n$ to zero (which is equivalent to $\delta_i = 1$) implies that the intended output falls to zero. But this is not optimal in the context of problem (2), which searches for efficiency improvements and hence does not allow the intended output level to fall below its initial level $\bar{y}$. Parts (b) and (c) of Proposition 5 state that, at a solution to problem (2), the optimal values of the proportional changes in the intended output and emission are given, respectively, by functions $\Theta$ and $\Gamma$ evaluated at the optimal vector of proportional changes in inputs, $\langle \bar{\delta}_o, \bar{\delta}_z\rangle$.[34] Furthermore, since problem (2) does not permit a fall in the good output, $\bar{\theta} = \Theta(\bar{\delta}_o, \bar{\delta}_z)$ cannot be negative. From the image of function $\Theta$ provided in (12) and (11), this implies that the optimal vector of proportional changes in inputs $\langle \bar{\delta}_o, \bar{\delta}_z\rangle$ must lie in $L_\delta$.

[34] For example, if $\bar{\theta}$ was not equal to $\Theta(\bar{\delta}_o, \bar{\delta}_z)$ then, since $\Theta(\bar{\delta}_o, \bar{\delta}_z)$ is the maximum proportional increase in the intended output starting from $\bar{y}$ when the proportional decrease in inputs is $\langle \bar{\delta}_o, \bar{\delta}_z\rangle$, it must be the case that $\bar{\theta} < \Theta(\bar{\delta}_o, \bar{\delta}_z)$. But this implies $\bar{s} = \langle \bar{\delta}_o, \bar{\delta}_z, \bar{\theta}, \bar{\gamma}\rangle$ is not an optimal solution to problem (2), as replacing $\bar{\theta}$ in $\bar{s}$ by $\Theta(\bar{\delta}_o, \bar{\delta}_z)$ is feasible and it increases the value of the objective function $W$ of this problem, which is defined as the weighted sum of proportional changes in inputs and outputs.

## 5.1 Computing the weighted index of graph inefficiency/efficiency improvements using functions Θ and Γ.

Proposition (6) above and the functional representation of $L_\delta$ in (10) imply that problem (2) can be re-written as[35]

$$\begin{aligned} \mathcal{I}^G(\bar{x}_o, \bar{x}_z, \bar{y}, \bar{z}; w) \quad &= \quad \max_{\delta_o, \delta_z} W(\delta_o, \delta_z, \Theta(\delta_o, \delta_z), \Gamma(\delta_z); w) \\ & \quad\quad \text{subject to} \\ \langle \delta_o, \delta_z \rangle \in L_\delta \quad &\Longleftrightarrow \quad \bar{y} \leq F(\bar{x}_o - (\delta_o \otimes \bar{x}_o), \ \bar{x}_z - (\delta_z \otimes \bar{x}_z)) \text{ and } \langle \delta_o, \delta_z \rangle \in [0,1]^n. \quad (14) \end{aligned}$$

Problem (14) replaces $\theta$ and $\gamma$ in problem (2) by functions Θ and Γ and requires that the vector of proportional changes in inputs lead to points in the input requirement set of $\bar{y}$, *i.e.*, $\langle \delta_o, \delta_z \rangle \in L_\delta$. The Lagrangian of the problem is

$$\mathcal{L} = W(\delta_o, \delta_z, \Theta(\delta_o, \delta_z), \Gamma(\delta_z); w) - \lambda [\bar{y} - F(\bar{x}_o - (\delta_o \otimes \bar{x}_o), \ \bar{x}_z - (\delta_z \otimes \bar{x}_z))].$$

## 5.2 First-order conditions and their interpretations.

In order to understand the economic intuition underlying the solution to the problem (14), in this section, we consider the case where Θ and Γ are continuously differentiable and adopt standard calculus techniques to characterise the solution. Noting from Proposition 6 that, due to essentiality of inputs, the upper bound of one on inputs is non-binding, the following Kuhn-Tucker first-order conditions (FOCs) hold at a solution $\langle \bar{\delta}_o, \bar{\delta}_z \rangle$ to the problem.

$$w^x_{o_i} \leq -w^y \frac{\partial \Theta}{\partial \delta_{o_i}} + \bar{\lambda} \frac{\partial F}{\partial x_{o_i}} x_{o_i}; \quad \bar{\delta}_{o_i} \geq 0; \quad \bar{\delta}_{o_i} \left[ w^x_{o_i} + w^y \frac{\partial \Theta}{\partial \delta_{o_i}} + \bar{\lambda} \frac{\partial F}{\partial x_{o_i}} x_{o_i} \right] = 0 \quad \forall\, i = 1, \ldots, n_o \quad (15)$$

$$w^x_{z_i} + w^z \frac{\partial \Gamma}{\partial \delta_{z_i}} \leq -w^y \frac{\partial \Theta}{\partial \delta_{z_i}} + \bar{\lambda} \frac{\partial F}{\partial x_{z_i}} x_{z_i}; \ \bar{\delta}_{z_i} \geq 0; \ \bar{\delta}_{z_i} \left[ w^x_{z_i} + w^z \frac{\partial \Gamma}{\partial \delta_{z_i}} + w^y \frac{\partial \Theta}{\partial \delta_{z_i}} + \bar{\lambda} \frac{\partial F}{\partial x_{z_i}} x_{z_i} \right] = 0 \quad (16)$$
$$\forall\, i = 1, \ldots, n_z$$

[35]Note that the objective function is

$$W(\delta_o, \delta_z, \Theta(\delta_o, \delta_z), \Gamma(\delta_z); w) = \sum_{i=1}^{n_o} w^x_{o_i} \delta_{o_i} + \sum_{i=1}^{n_z} w^x_{z_i} \delta_{z_i} + w^y \Theta(\delta_o, \delta_z) + w^z \Gamma(\delta_z).$$

$$F\left(\bar{x}_o - \left(\bar{\delta}_o \otimes \bar{x}_o\right),\ \bar{x}_z - \left(\bar{\delta}_z \otimes \bar{x}_z\right)\right) \geq \bar{y}, \quad \bar{\lambda} \geq 0,$$
$$\bar{\lambda}\left[\bar{y} - F\left(\bar{x}_o - \left(\bar{\delta}_o \otimes \bar{x}_o\right),\ \bar{x}_z - \left(\bar{\delta}_z \otimes \bar{x}_z\right)\right)\right] = 0 \tag{17}$$

From FOC (17), it is clear that, if the Lagrange multiplier takes a positive value, *i.e.,* $\bar{\lambda} > 0$, then the constraint in problem (14) is binding (holds as an equality). This means that the solution $\langle \bar{\delta}_o, \bar{\delta}_z \rangle$ lies in $I_\delta$ and hence leads to points in the isoquant of $\bar{y}$. As seen in (12), the image of function $\Theta$ implies that, in this case, the optimal proportionate change in the intended output is zero, *i.e.,* $\bar{\theta} = \Theta\left(\bar{\delta}_o, \bar{\delta}_z\right) = 0$.

On the other hand, if the solution leads to a point in the input requirement set of $\bar{y}$ but not in its isoquant, *i.e.,* if $\langle \bar{\delta}_o, \bar{\delta}_z \rangle \in L_\delta \setminus I_\delta$, then the constraint of problem (14) holds as a strict inequality and (i) the FOC (17) implies that the Lagrange multiplier takes a value $\bar{\lambda} = 0$ and (ii) the image of function $\Theta$ in (12) implies that the optimal proportionate change in the intended output is bigger than zero, *i.e.,* $\bar{\theta} = \Theta\left(\bar{\delta}_o, \bar{\delta}_z\right) > 0$.

**Remark 7** *We can interpret the terms of the FOCs as (15) and (16) as follows:*

*(a) The term $w^x_{o_i}$ on the left-side of the first inequality in FOC (15) is the direct marginal gain in the weighted sum of proportional changes $W$ due to a marginal increase in $\delta_{o_i}$.*

*(b) The right-side of the first inequality in FOC (15) consists of (i) the indirect marginal loss in $W$ induced by a marginal increase in $\delta_{o_i}$ (given by $-w^y \frac{\partial \Theta}{\partial \delta_{o_i}}$) and (ii) the adjustment costs incurred in meeting the constraint of problem (14) when $\delta_{o_i}$ changes (given by $\bar{\lambda}\frac{\partial F}{\partial x_{o_i}}$). The loss in (i) occurs because a marginal increase in $\delta_{o_i}$ decreases $W$ by decreasing $\Theta$.*

*(c) The term on the left-side of the first inequality in FOC (16) is the total marginal gain in $W$ due to a marginal increase in $\delta_{z_i}$. This includes (i) a direct gain $w^x_{z_i}$ and (ii) an indirect gain $w^z \frac{\partial \Gamma}{\partial \delta_{z_i}}$. The latter can be interpreted as the marginal gain in $W$ due to an increase in $\Gamma$ induced by the marginal increase in $\delta_{z_i}$.*

*(d) The interpretation of the right-side of the first inequality in FOC (16) is exactly similar to the interpretation given in (b) above. It is the additional cost incurred due to a marginal increase in $\delta_{z_i}$*

The above remark implies that the computation of the weighted index of inefficiency/ efficiency improvements $\mathcal{I}^G$ involves both marginal gains and losses. An increase in $\delta_i$ (which is equivalent to a proportional reduction in the usage of the $i^{th}$ input) implies a marginal gain in the weighted sum of proportional changes $W$ (as described in parts (a) and (c)). On the other hand, it also involves some marginal costs (as described in parts (b) and (d)). At an interior optimal value of $\delta_i$, *i.e.,* when $\bar{\delta}_i > 0$ for $i = 1, \ldots, n$, the FOCs (15) and (16) imply that these marginal gains and losses are equalised. At a corner optimum, *i.e.,* when $\bar{\delta}_i = 0$, the marginal losses of further proportional reduction in the $i^{th}$ input are bigger than the marginal gains.

In particular, starting from $\bar{x}_z$, reducing the usage of $i^{th}$ emission-causing input ($i.e.,\ \bar{\delta}_{z_i} > 0$) leads to a trade-off between efficiency improvement along the intended output direction and efficiency improvement along the emission direction. This is because, such a reduction has two consequences. On the one hand, it implies a reduction in $\Theta$, which measures the maximum possible proportional increase in the intended output. On the other hand, it also implies an increase in $\Gamma$, which measures the maximum possible proportional reduction in emission. These consequences of reducing an emission-causing input starting from $\bar{x}_z$ can clearly be seen in panels (a) and (b) of Figure 1. When the usage of the emission-causing input is reduced to promote efficiency improvement in the input direction, an increase in environmental efficiency improvement $\gamma$ comes at the cost of reduction in the efficiency improvement along the intended output direction $\theta$.

# 6 Characterising all possible solutions.

Table 1 characterises all the possible solutions of problem (2) (or equivalently, problem (14)).[36] If $\bar{s} = \langle \bar{\delta}_o, \bar{\delta}_z, \bar{\theta}, \bar{\gamma} \rangle$ denotes a solution of problem (2), then it follows from Proposition 6 that $\bar{\theta} = \Theta\left(\bar{\delta}_o, \bar{\delta}_z\right)$ and $\bar{\gamma} = \Theta\left(\bar{\delta}_z\right)$. The columns of Table 1 cover the possible values $\bar{\gamma}$ can take given the image of function $\Gamma$ defined in (13), while its rows cover the possible values $\bar{\theta}$ can take given the image of function $\Theta$ defined in (12). For each combination of values of $\bar{\gamma}$ and $\bar{\theta}$, the table shows the possible values $\langle \bar{\delta}_o, \bar{\delta}_z \rangle$ can take such that $\bar{\theta} = \Theta\left(\bar{\delta}_o, \bar{\delta}_z\right)$ and $\bar{\gamma} = \Gamma\left(\bar{\delta}_z\right)$.

## 6.1 Rows of Table 1.

It is clear from (12) that the maximum value that function $\Theta$ can take is $\bar{\beta}_y - 1$, which happens when $\delta = 0_n$. This measures the (optimal) efficiency improvement in the intended output direction when all inputs are held fixed (see also panel (a) of Figure 2).

Rows **(1)** and **(2)** of Table 1 correspond to $\bar{\theta}$ taking values smaller than $\bar{\beta}_y - 1$. Since $\Theta\left(0_n\right) = \bar{\beta}_y - 1$ and because $\Theta$ is a non-increasing function of $\delta$ (see Theorem 5), the optimum of problem (2) (equivalently, problem (14)), when positioned in Rows **(1)** and **(2)** of Table 1, necessarily requires reductions in inputs, *i.e.,* $\bar{\delta} > 0_n$ when $\bar{\theta} < \bar{\beta}_y - 1$. The differences in the situations covered by Rows **(1)** and **(2)** are described below:

- In Row **(1)**, $\bar{\theta}$ takes a value strictly greater than zero. Thus, the optimum recommends efficiency improvement in the intended output direction, in addition to efficiency im-

[36] It covers the general case where $\Theta$ and $\Gamma$ need not be differentiable.

**Table 1**

| | $\bar{\gamma} > 1 - \bar{\beta}_z \geq 0$ **(1)** | $\bar{\gamma} = 1 - \bar{\beta}_z > 0$ **(2)** | $\bar{\gamma} = 1 - \bar{\beta}_z = 0$ **(3)** |
|---|---|---|---|
| $0 < \bar{\theta} < \bar{\beta}_y - 1$ **(1)** | $L_\delta \setminus (I_\delta \cup \{0_n\})$ <br> $\bar{\delta}_z > 0_{n_z}$ | $L_\delta \setminus (I_\delta \cup \{0_n\})$ <br> $\bar{\delta}_z = 0_{n_z}$ or $\bar{\delta}_z > 0_{n_z}{}^{\dagger}$ | $L_\delta \setminus (I_\delta \cup \{0_n\})$ <br> $\bar{\delta}_z = 0_{n_z}{}^{\spadesuit}$ or $\bar{\delta}_z > 0_{n_z}{}^{\dagger}$ |
| $0 = \bar{\theta} < \bar{\beta}_y - 1$ **(2)** | $0_n < \bar{\delta} \in I_\delta$ <br> $\bar{\delta}_z > 0_{n_z}$ | $0_n < \bar{\delta} \in I_\delta$ <br> $\bar{\delta}_z = 0_{n_z}$ or $\bar{\delta}_z > 0_{n_z}{}^{\dagger}$ | $0_n < \bar{\delta} \in I_\delta$ <br> $\bar{\delta}_z = 0_{n_z}{}^{\spadesuit}$ or $\bar{\delta}_z > 0_{n_z}{}^{\dagger}$ |
| $\bar{\theta} = \bar{\beta}_y - 1 > 0$ **(3)** | $\bar{\delta} = 0_n$ or <br> $\bar{\delta} \in L_\delta \setminus (I_\delta \cup \{0_n\})^*$ <br> $\bar{\delta}_z > 0_{n_z}$ | $\bar{\delta} = 0_n$ or <br> $\bar{\delta} \in L_\delta \setminus (I_\delta \cup \{0_n\})^*$ <br> $\bar{\delta}_z = 0_{n_z}$ or $\bar{\delta}_z > 0_{n_z}{}^{\dagger}$ | $\bar{\delta} = 0_n$ or <br> $\bar{\delta} \in L_\delta \setminus (I_\delta \cup \{0_n\})^*$ <br> $\bar{\delta}_z = 0_{n_z}{}^{\spadesuit}$ or $\bar{\delta}_z > 0_{n_z}{}^{\dagger}$ |
| $\bar{\theta} = \bar{\beta}_y - 1 = 0$ **(4)** | $\bar{\delta} = 0_n{}^{\clubsuit}$ or <br> $\bar{\delta} \in L_\delta \setminus (I_\delta \cup \{0_n\})^*$ <br> $\bar{\delta}_z > 0_{n_z}$ | $\bar{\delta} = 0_n{}^{\clubsuit}$ or <br> $\bar{\delta} \in L_\delta \setminus (I_\delta \cup \{0_n\})^*$ <br> $\bar{\delta}_z = 0_{n_z}$ or $\bar{\delta}_z > 0_{n_z}{}^{\dagger}$ | $\bar{\delta} = 0_n{}^{\clubsuit}$ or <br> $\bar{\delta} \in L_\delta \setminus (I_\delta \cup \{0_n\})^*$ <br> $\bar{\delta}_z = 0_{n_z}{}^{\spadesuit}$ or $\bar{\delta}_z > 0_{n_z}{}^{\dagger}$ |

$^{\dagger}$ $\Gamma(\delta_z) = 1 - \bar{\beta}_z$ for all $\delta_z$ such that $\bar{\delta}_z \geq \delta_z \geq 0_{n_z}$.
$^{\spadesuit}$ $\langle \bar{x}_o, \bar{x}_z, \bar{y}, \bar{z} \rangle$ lies on the lower frontier of $T_2$.
$^{*}$ $\Theta(\delta) = \bar{\beta}_y - 1$ for all $\delta$ such that $\bar{\delta} \geq \delta \geq 0_n$.
$^{\clubsuit}$ $\langle \bar{x}_o, \bar{x}_z, \bar{y}, \bar{z} \rangle$ is an efficient point of sub-technology $T_1$.
When $^{\spadesuit}$ and $^{\clubsuit}$ are simultaneously true then $\langle \bar{x}_o, \bar{x}_z, \bar{y}, \bar{z} \rangle$ is an efficient point of technology $T$.

provements in the input directions, *i.e.,* the optimum involves proportional increase in the intended output as well as proportional reductions in inputs.

- In Row **(2)**, $\bar{\theta}$, the optimal proportional increase in the intended output, takes a value exactly equal to zero. Thus, this is a case where there are efficiency improvements only in the input direction, with no change in the level of the intended output produced, *i.e.,* the optimum recommends producing the original level of intended output $\bar{y}$ with the greatest feasible reductions in the inputs. Clearly, this is a case, where the reduced input vector will lie in the isoquant of $\bar{y}$, *i.e.,* $\bar{\delta} \in I_\delta \setminus \{0_n\}$.

Rows **(3)** and **(4)** of Table 1 cover the cases where $\bar{\theta}$ is exactly equal to $\bar{\beta}_y - 1$, which is the optimal proportional increase in the intended output when all inputs are held fixed at the original level $\bar{x}$.

- In Row **(3)**, $\bar{\theta} = \bar{\beta}_y - 1$ takes a value greater than zero, indicating that there is output-based productive inefficiency at the initial production vector $\bar{v}$, *i.e.,* $\bar{v}$ is not on the weakly efficient frontier of sub-technology $T_1$.

- In Row **(4)**, $\bar{\theta} = \bar{\beta}_y - 1$ is equal to zero. Since $\bar{\beta}_y$ is exactly equal to one in this case, there is no output-based productive inefficiency at the initial production vector $\bar{v}$, which is hence on the weakly efficient frontier of sub-technology $T_1$. (See Remark 4.)

In both Rows **(3)** and **(4)** of Table 1, two possibilities arise with respect to optimal reductions in the inputs:

(i) Since $\Theta(0_n) = \bar{\beta}_y - 1$, a possible recommendation of the optimum of problem (2) (and problem (14)) is not to reduce usage of any input, *i.e.,* $\bar{\delta} = 0_n$. Two cases arise: (a) In Row **(3)**, $\bar{\beta}_y - 1 > 0$, so that the optimum will involve an efficiency improvement of magnitude $\bar{\theta} = \bar{\beta}_y - 1 > 0$ along the intended output dimension. (b) In Row **(4)**, $1 - \bar{\beta}_y$ is equal to zero. Hence, there is no scope for efficiency improvements along both the intended output and input dimensions at the original production vector $\bar{v}$. Hence, $\bar{v}$ is an efficient point of sub-technology $T_1$. (See Remark 4.)

(ii) It is also possible that the optimum recommends reductions in inputs, *i.e.,* $\bar{\delta} > 0_n$, even when $\bar{\theta} = \bar{\beta}_y - 1$. This is true when inputs have been employed in excessive amounts at the original production vector $\bar{v}$. In that case, starting from $\bar{v}$, the optimum of problem (2) recommends efficiency improvements (reductions) along the input dimensions. This is illustrated in panels (a) and (b) of Figure 4, which assumes $n = n_z = 1$. In both panels,

$\bar{\delta} > 0$, so that the new optimal level of input is $x^* = \bar{x} - \bar{\delta}\bar{x}$ which is less than $\bar{x}$. Panel (a) shows that $\bar{\theta} = \bar{\beta}_y - 1 > 0$, while panel (b) shows that $\bar{\theta} = \bar{\beta}_y - 1 = 0$.[37]

### 6.2 Columns of Table 1.

The image of $\Gamma$ in (13) implies that the minimum value it can take is $1 - \bar{\beta}_z$, which happens when $\delta_z = 0_{n_z}$. This measures the output-based environmental efficiency improvement, *i.e.,* it is the optimal proportional reduction in emission when all emission-causing inputs are held fixed at the initial level $\bar{x}_z$.

Column **(1)** of Table 1 corresponds to $\bar{\gamma}$ taking a value bigger than $1 - \bar{\beta}_z$. Since $\Gamma\left(0_{n_z}\right) = 1 - \bar{\beta}_z$ and because $\Gamma$ is a non-decreasing function of $\delta_z$, the optimum to problem (2) necessarily involves a reduction in the usage of emission-causing inputs, *i.e.,* $\bar{\delta}_z > 0_{n_z}$. Further, the total efficiency improvement in the emission direction at the optimum of problem (2) can be decomposed into

$$\bar{\gamma} \;\; = \;\; \left[1 - \bar{\beta}_z\right] \; + \; \left[\bar{\gamma} - (1 - \bar{\beta}_z)\right] \tag{18}$$

where $1 - \bar{\beta}_z$ is the maximum proportional reduction in emission that is possible when emission-causing inputs are held fixed, while $\left[\bar{\gamma} - (1 - \bar{\beta}_z)\right]$ is the proportional reduction in emission that is attributable purely to reduction in the usage of the emission-causing inputs.

Columns **(2)** and **(3)** of Table 1 correspond to $\bar{\gamma}$ taking a value exactly equal to $1 - \bar{\beta}_z$.

- In Column **(2)**, $1 - \bar{\beta}_z$ takes a value greater than zero, indicating that there is output-based environmental inefficiency at the initial production vector $\bar{v}$, *i.e.,* $\bar{v}$ is *not* on the lower frontier of sub-technology $T_2$.

- In Column **(3)**, $1 - \bar{\beta}_z$ takes a value exactly equal to zero, indicating that there is no output-based environmental inefficiency at the initial production vector $\bar{v}$, *i.e.,* $\bar{v}$ is on the lower frontier of sub-technology $T_2$. (See Remark 4.)

[37]Note that, since $\Theta$ is a non-increasing function and $\Theta(0_n) = \bar{\beta}_y - 1$, we have $\Theta\left(\delta\right) = \bar{\beta}_y - 1$ for all $\delta$ such that $\bar{\delta} \geq \delta \geq 0_n$. This is seen in both panels (a) and (b) of Figure 4.

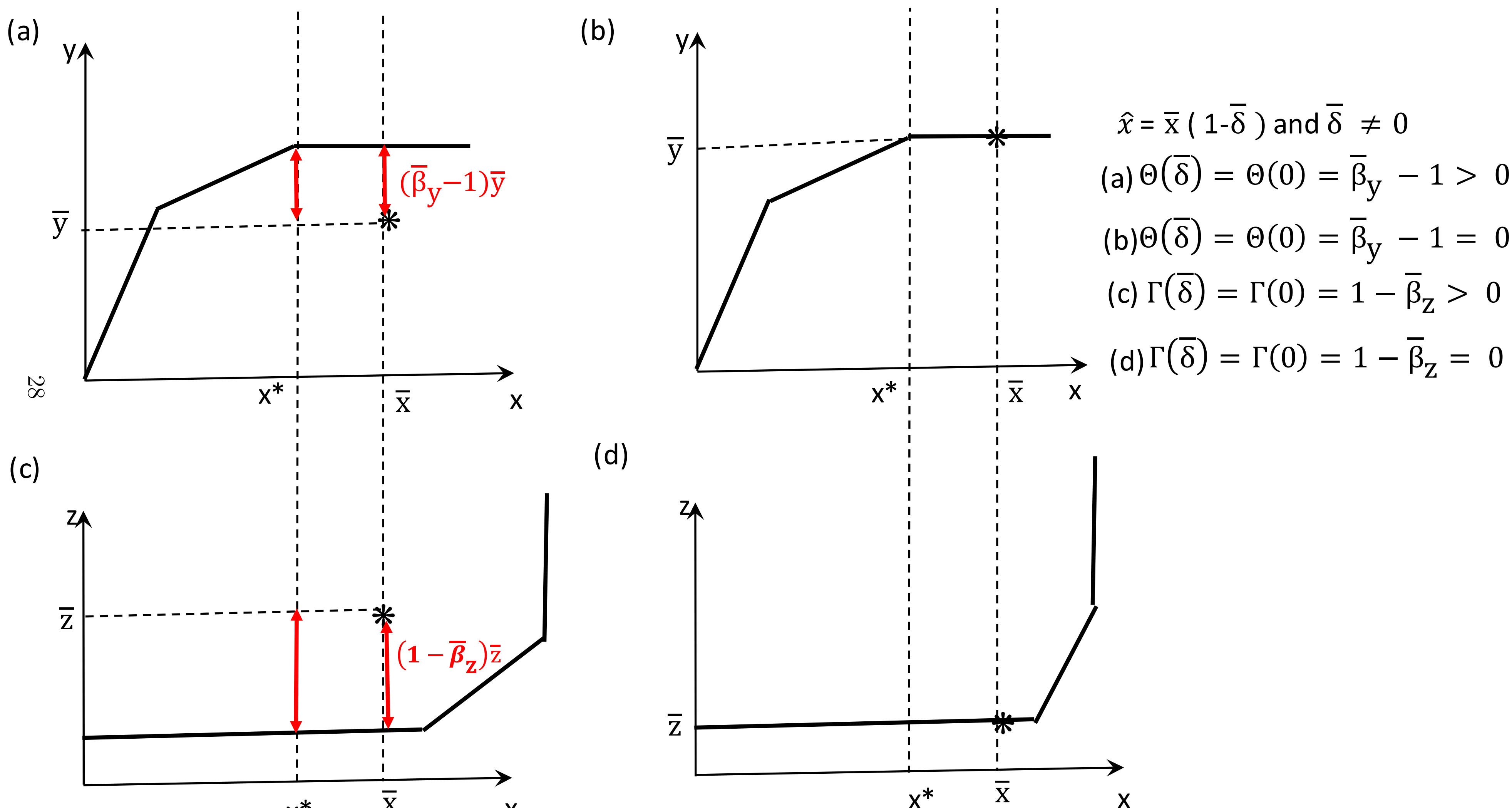
(a)
(b)
(c)
(d)
y
z
x
x*
$\bar{x}$
$\bar{y}$
$\bar{z}$
$(\bar{\beta}_y - 1)\bar{y}$
$(1 - \bar{\beta}_z)\bar{z}$
$\hat{x} = \bar{x}\,(1-\bar{\delta})$ and $\bar{\delta} \neq 0$
(a) $\Theta(\bar{\delta}) = \Theta(0) = \bar{\beta}_y - 1 > 0$
(b) $\Theta(\bar{\delta}) = \Theta(0) = \bar{\beta}_y - 1 = 0$
(c) $\Gamma(\bar{\delta}) = \Gamma(0) = 1 - \bar{\beta}_z > 0$
(d) $\Gamma(\bar{\delta}) = \Gamma(0) = 1 - \bar{\beta}_z = 0$

Figure 4

In both Columns **(2)** and **(3)**, two possibilities arise with respect to optimal reductions of emission-causing inputs:

(i) Since $\Gamma(0_{n_z}) = 1 - \bar{\beta}_z$, one possibility is that there is no change in the usage of emission-causing inputs at the optimum, *i.e.,* $\bar{\delta}_z = 0_{n_z}$. In this case, if $1 - \bar{\beta}_z > 0$ (the case of Column **(2)**) then the optimum will also require a proportional reduction in the emission level that is equal to $1 - \bar{\beta}_z$. On the other hand, if $1 - \bar{\beta}_z = 0$ (the case of Column **(3)**) then there is no scope at all for reduction in emission at the optimum.

(ii) It is also possible that the optimum recommends proportional reductions in the usage of emission-causing inputs ($\bar{\delta}_z > 0_{n_z}$). This is illustrated in panels (c) and (d) of Figure 4, where $\bar{\delta}_z > 0_{n_z}$. Panel (c) covers the case of Column **(2)**, where a proportional reduction in emission to the tune of $\bar{\gamma} = 1 - \bar{\beta}_z > 0$ is also required. Panel (d) covers the case of Column **(3)**, where $\bar{\gamma} = 1 - \bar{\beta}_z = 0$ and no further proportional reduction in emission is possible.[38]

### 6.3 Consequences of zero weights on inputs.

It is to be noted that assigning zero weights on inputs in problems (2) or (14) is not equivalent to computing output based inefficiency, where all inputs are held fixed.

To see the consequences of this restriction, note that FOC (15) and its interpretations in Remark 7 imply that, when a zero weight is attached to a non-emission causing input, then there is no gain/increase in the value of the objective function $W$ of problem (2) (equivalently, problem (14)) when this input is reduced.[39] On the contrary, a reduction in this input could impose a cost as it could reduce the scope of efficiency improvement in the intended-output direction (*i.e.,* the value taken by $\Theta$ reduces), thereby reducing $W$.[40] As, a result, the optimum of problem (2) (equivalently, problem (14)) recommends no changes in these inputs when zero weights are attached to them in the objective function.

On the other hand, FOC (16) and its interpretation in Remark 7 implies that the optimal proportional reduction in the emission-causing input may not be zero even when a zero weight is attached to it in the objective function $W$. This is because, although a reduction in this input may not imply a direct gain in $W$ on account of its zero weight, it does warranty an efficiency

[38] Note that, since $\Gamma$ is a non-decreasing function and $\Gamma(0_n) = 1 - \bar{\beta}_z$, we have $\Gamma(\delta_z) = \bar{1} - \beta_z$ for all $\delta$ such that $\bar{\delta} \geq \delta \geq 0_n$. This is seen in both panels (c) and (d) of Figure 4.

[39] Recall that function $W$ is defined as the weighted sum of proportional reductions in all inputs and emission levels and proportional increase in the intended output production. Also, recall that changes in non-emission causing inputs only influence the intended output and do not affect emission generation under condition (i') in the Definition 2 of a BPT.

[40] Recall from Theorem 5 that $\Theta$ is non-increasing in $\delta$, the vector of proportional reductions in inputs.

improvement (*i.e.,* a proportional reduction) in the emission direction (given by function $\Gamma$) through relations defining sub-technology $T_2$. It can hence lead to a gain in the objective function $W$ provided a positive weight is attached to $\gamma$ (which denotes the proportional reduction in emission) in the objective function. At the optimum, it is possible that this gain can offset any loss due to decline in the efficiency improvement in the intended output direction, given by function $\Theta$, that is also caused by the reduction in this input.

**Remark 8** *If the weight given to efficiency improvement in any non-emission causing input in problem (14) is zero then, at the optimum of problem (14), there is no change in the usage of this input, i.e., we have a corner solution for that input ($\bar{\delta}_{o_i} = 0$ whenever $w^x_{o_i} = 0$ for any $i = 1, \ldots, n_o$).*
*However, a zero weight given to efficiency improvement in any emission-causing input need not imply a corner solution for that input.*

# 7 Relation between efficiency improvements and thermodynamic efficiency.

In our empirical analysis, we study efficiency improvements in coal-based thermal power plants, where coal measured in heat units is the single emission-causing input. In engineering science, thermodynamic efficiency (TDE) of a thermal power plant is defined as the electrical output produced per unit of heat employed by the plant. Engineers of thermal power plants aim to increase the thermodynamic efficiency of electricity generation and, at the same time, minimise emission generation per unit of electricity generated (denoted by EPUE).[41] Recalling that amount of coal employed ($x_z$) is measured in heat units, we can define thermodynamic efficiency as

$$TDE\,(x_o, x_z, y, z) \;=\; \frac{y}{x_z} \qquad \text{and} \qquad EPUE\,(x_o, x_z, y, z) \;=\; \frac{z}{y}$$

Then the percentage increase in TDE is given by the difference between the percentage increase in electrical output and the percentage increase in coal. Similarly, we can define the percentage reduction in EPUE.

$$\frac{d\,TDE}{TDE} \;=\; \frac{dy}{y} - \frac{dx_z}{x_z} \qquad \text{and} \qquad -\frac{d\,EPUE}{EPUE} \;=\; -\frac{dz}{z} + \frac{dy}{y}.$$

In general, increases in TDE are consistent with no efficiency improvements in the electricity

[41]See the report of IEA's Coal Industry Advisory Board (2010).

or coal input directions, *i.e.,* with $\frac{dy}{y}$ taking negative values or $\frac{dx_z}{x_z}$ taking positive values. Similarly, reductions in EPUE are consistent with no efficiency improvements in the emission or electricity directions, *i.e.,* with $\frac{dz}{z}$ taking positive values or $\frac{dy}{y}$ taking negative values. However, in problem (2), $\theta$ denotes the percentage *increase* in the intended output, while $\delta_z$ and $\gamma$ denote the percentage *reductions* in the emission-causing input and emission, respectively. Hence, $\frac{dx_z}{x_z} = -\delta_z$ and $\frac{dz}{z} = -\gamma$. Since the problem is concerned with designing improvements in technical efficiency, $\theta$, $\delta_z$, and $\gamma$ are restricted to taking non-negative values. This implies that the technical efficiency improvements studied in this paper always imply increases in TDE and reductions in EPUE of power plants.

$$\frac{dTDE}{TDE} = \theta + \delta_z \geq 0 \qquad \text{and} \qquad -\frac{d\ EPUE}{EPUE} = \gamma + \theta \geq 0. \tag{19}$$

In our empirical analysis, we will study the improvements in TDE and reductions in EPUE implied by optimal configurations of efficiency improvements.

# 8 Data and methodology.

## 8.1 Data.

This study uses data on 47 coal-fired thermal power plants in India for the year 2014. The data was collected from the annual publication of the Central Electricity Authority (CEA) of India (2013-14, 2014-15). The plants studied are run by 16 major power generating companies operating in various states of India.

The intended output of the power plants is net electricity, which is measured in gigawatt hours (GWh). Since data on capital and labour employed in these power plants is not available, plant capacity, measured in megawatt (MW), is used as a proxy for capital,[42] while labour is not included as an input.[43] Aggregate heat from coal and oil consumption by the coal-based thermal power plants measured in millions of kilocalories (mill of Kcal) is taken as the emission-causing input (fossil fuel input).[44] The model also employs CEA data on plant operating availability as a managerial input.[45] It is the percentage of total capacity (measured in MWh) that is

[42]A similar approach is also taken in other recent works on Indian thermal power plants (see *e.g.,* Sahoo et al. (2017) and Behera et al., (2010)).

[43]It has been argued that the contribution of labour cost to total operating costs of these power plants is very small (see *e.g.,* Kumar et al (2015)).

[44]Oil is the secondary fuel in coal-based thermal power plants. It is employed primarily to cover the start-up fuel requirements and for flame stabilisation, and does not contribute significantly to electricity generation in these power plants.

[45]Many recent works on thermal power plants such as Sueyoshi and Goto (2010, 2011, 2012) and Sahoo et al. (2017) include managerial inputs.

available to the plant for electricity generation after subtracting out the percentage lost due to forced outage and planned maintenance. Finally, data on $CO_2$ emission (the bad output, which is measured in metric tons (Mtons)) is generated by the CEA using a linear formula that employs a constant average emission and oxidation factors across all grades of coal and oil.

For greater details on the dataset, the reader is referred to Murty and Nagpal (2018).

## 8.2 Methodology.

A DEA methodology is employed in this paper to estimate the weighted index of graph inefficiency/efficiency improvements of plants in the Indian coal-based thermal power sector. Thermal power plants in our dataset are indexed by $u = 1, \ldots, U$, where $U = 47$. The description of our dataset implies that, in our study, $n_o = 2$, while $n_z = 1$, so that the number of inputs is $n = n_o + n_z = 3$. The $U \times n$-dimensional data matrix of inputs is denoted by

$$X = \begin{bmatrix} X_o & X_z \end{bmatrix},$$

where $X_o$ and $X_z$ are, respectively, the $U \times n_o$ and $U \times n_z$-dimensional data matrices of non-emission causing and emission-causing inputs. The $U \times 1$-dimensional data matrices of intended output and emission are denoted by $Y$ and $Z$, respectively.

The DEA specification of a BPT is adapted from MRL as follows: First, the DEA specification of sub-technology $T_1$ is[46]

$$T_1 = \left\{ \langle x, y, z \rangle \in \mathbf{R}_+^{n+2} \;\middle|\; \lambda^\top X \leq x^\top,\ \lambda^\top Y \geq y,\ \lambda \geq 0_U \right\} \tag{20}$$

Since we are concerned with measuring technical efficiency, one part of which requires computing how far power plants are from the lower frontier of sub-technology $T_2$, it suffices to focus on its costly disposal hull $CDH\,(T_2)$. Under our maintained assumptions, the lower frontiers of sub-technology $T_2$ and set $CDH\,(T_2)$ are the same (see Definition 2 of a BPT) and set $T_2$ is a subset of set $CDH\,(T_2)$. The costly disposal hull of sub-technology $T_2$ has a convenient DEA representation in MRL:[47]

$$CDH\,(T_2) = \left\{ \langle x_o, x_z, y, z \rangle \in \mathbf{R}_+^{n+2} \;\middle|\; \mu^\top X_z \geq x_z^\top,\ \mu^\top Z \leq z,\ \mu \geq 0_U \right\} \tag{21}$$

[46]The vector $\lambda^\top$ denotes the transpose of a $U$ dimensional vector $\lambda$. $0_U$ is a $U$-dimensional zero vector.

[47]The vector $\mu^\top$ denotes the transpose of a $U$ dimensional vector $\mu$.

We define set $\bar{T}$ as the intersection of sub-technologies $T_1$ and $CDH$ $(T_2)$. Thus, it is given by:

$$\bar{T} = \Big\{ \langle x_o, x_z, y, z \rangle \in \mathbf{R}_+^{n+2} \;\Big|\; \lambda^\top X \leq x^\top, \quad \lambda^\top Y \geq y, \quad \mu^\top X_z \geq x_z^\top, \quad \mu^\top Z \leq z, \quad \lambda \geq 0_U, \quad \mu \geq 0_U \Big\}$$

The above specification implies that set $\bar{T}$ exhibits constant returns to scale, is convex, satisfies free input disposability of the non-emission causing inputs and free output disposability of the intended output, but is not freely disposable in the emission-causing inputs. It satisfies costly disposability with respect to the emission. Given its construction, the weakly efficient and the efficient frontiers of $\bar{T}$ will be the same as those of the original overall technology $T = T_1 \cap T_2$.

The DEA programme for computing the weighted index of graph efficiency improvements for the $u^{th}$ thermal power plant in our data set is:

$$\begin{aligned} \mathcal{I}^G(x_o^u, x_z^u, y^u, z^u) \;=\; \max_{\delta,\theta,\gamma} \quad & w_{o_1}\delta_{o_1} + w_{o_2}\delta_{o_2} + w_z\delta_z + w_y\theta + w_z\gamma \\ & \text{subject to} \\ & \theta y^u - \lambda^\top Y \leq -y^u, \quad (\delta \otimes x^u)^\top + \lambda^\top X \leq x^{u\top}, \\ & -\delta_z x_z^u - \mu^\top X_z \leq -x_z^u, \quad \gamma z^u + \mu^\top Z \leq z^u, \quad \lambda \in \mathbf{R}_+^U, \quad \mu \in \mathbf{R}_+^U \end{aligned}$$

# 9 Results and interpretations.

Some special aspects of the results in Murty and Nagpal (2018) on the output-based FGL index computed for the Indian coal-based thermal power plants and its decomposition into productive and environmental efficiency were discussed in the introductory section (Section 1) of this paper. Table 2 corroborates these points. It shows that, in comparison to the output-based measure of productive efficiency improvement $\bar{\beta}_y - 1$, the output-based measure of environmental efficiency $1 - \bar{\beta}_z$ has very low variability, as measured by the standard deviation. Moreover, the average and maximum values of the latter are much lower (and close to zero) as compared to those of the former.

In our empirical analysis, we address the question whether there can be significant improvements in environmental and productive efficiency across plants in the Indian coal-based thermal power sector, when the emission-causing input is also allowed to adjust efficiently along with both the good and the bad outputs. We find that the nature and the extent of these efficiency improvements (i) depend crucially on the weights chosen while computing the weighted graph index of efficiency improvements $\mathcal{I}^G$ and (ii) are highly correlated to the output-based measure of productive efficiency $\mathbf{b}_y$ measured in Murty and Nagpal (2018). Table 2 provides the descriptive statistics of $\mathbf{b}_y$.

**Table 2: Descriptive statistics**

| | $\mathbf{b}_y$ | $\beta_y$-1 | 1-$\beta_z$ |
|---|---|---|---|
| Avg | 0.833 | 0.276 | 0.100 |
| Max | 1 | 2.712 | 0.181 |
| Min | 0.269 | 0 | 0.000 |
| Std dev | 0.160 | 0.442 | 0.027 |

Hence, in our analysis, we group all the coal-based power plants in our data set into three categories based on the values taken by the output-based productive efficiency index $\mathbf{b}_y$ across these plants: (1) high performers, where $\mathbf{b}_y$ ranges between 0.95 to 1; (2) moderate performers, where $\mathbf{b}_y$ ranges between 0.795 to 0.95; and (3) low performers, where $\mathbf{b}_y$ ranges between 0 to 0.795. We now study the differences in efficiency improvements recommended by the optimum of problem (2) across these three types of performers for various weighting schemes.

To focus sharply on studying the extent of efficiency improvements possible in the good and bad output directions and the trade-offs between these two efficiency improvements, in our empirical analysis, we restrict ourselves to cases where zero weights are assigned to all inputs.[48] As a result, the optimum recommends no change in the non-emission causing inputs ($\bar{\delta}_o = 0_{n_o}$). On the other hand, as stated in Remark 8, the optimum may recommend positive reductions in the usage of coal even when a zero weight is assigned to it in problem (2).

## 9.1 Equal weights on $\theta$ and $\gamma$ and zero weight on $\delta_z$.

Table 3 summarises the results in the case when equal weights are assigned to efficiency improvements along the intended output and the bad output directions, which are given by $\theta$ and $\gamma$, respectively (*i.e.,* $w_y = w_z = 0.5$), while a zero weight is assigned to the efficiency improvement in the emission-causing input direction, given by $\delta_z$ (*i.e.,* $w_{x_z} = 0$). Table 3 lists all the positions (solution categories) in Table 1, where the solutions of problem (2) are observed for different plants. For example, position $(4, 2)$ denotes the fourth row and second column of Table 1. Only five out of the twelve positions in Table 1 are observed in this weighting scheme.[49]

### 9.1.1 Optimal efficiency improvements in the case of the high performers.

It is clear that, for plants with high values of output-based productive efficiency $\mathbf{b}_y$, the scope for efficiency improvement in the intended output direction is limited, when the inputs are held

[48] To see the implications of this restriction, see also Section 6.3 and Remark 8.

[49] Refer to Table 1 and its detailed explanation in Section 6. Note that only a subset of all the possible positions in Table 1 will be observed under any given weighting scheme.

fixed.[50] But Table 3 reveals that, when the emission-causing input is also allowed to vary, the efficiency improvement in its direction is also very small for plants in this category, *i.e.,* the optimum of problem (2) involves very small proportional reductions in coal for these plants: $\delta_z$ ranges between 0% and 7% with an average of 2%.

In particular, 4 out of the 14 plants in this category, namely, Bokaro B, Dahanu, Korba, and Vindhyanchal are positioned in Row **(4)** of Table 1. This implies that $\bar{\theta} = \bar{\beta}_y - 1 = 0$ for these plants, *i.e.,* they lie on the weakly efficient frontier of sub-technology $T_1$. (See Table 1 and its discussion in Section 6). In addition, we find that the optimal proportional reduction in coal ($\bar{\delta}_z$) is also zero for these plants. This implies that, for these plants, there is no scope for efficiency improvement in either the intended output or the emission-causing input directions, *i.e.,* the optimum for these plants lies also on the efficient frontier of sub-technology $T_1$. Since the optimum recommends no reduction in the usage of coal, the efficiency improvement in the emission direction, *i.e.,* the optimal proportional reduction in $CO_2$ ($\bar{\gamma}$), is given purely by the output-based measure $1 - \bar{\beta}_z$. This measures the maximum proportional reduction in emission when the emission-causing input is held fixed. Since the optima for these plants are positioned in Column **(2)** of Table 1, $1 - \bar{\beta}_z$ is positive. Thus, the output-based environmental efficiency index $\bar{\beta}_z$ is less than one, indicating that these plants do not lie on the lower frontier of sub-technology $T_2$. Nevertheless, as argued above and in the introductory section (Section 1) given the methodology employed by CEA to compute $CO_2$ emission data, all plants (including the high performers) operate very close to the lower frontier of sub-technology $T_2$, and hence $1 - \bar{\beta}_z$ is low for all plants in our dataset.[51] Hence, the optimal proportional reduction in $CO_2$ emission for these four high performers (which is given by $1 - \bar{\beta}_z$) though positive is very small.

Table 3 also shows that the optima for a majority of high performing plants (9 out of 14 plants) are positioned in Row **(2)** and Column **(1)** of Table 1. The former implies that, for these plants, the optimum recommends producing the initial level of electricity ($\bar{y}$) with least possible use of coal, *i.e.,* it recommends no efficiency improvement in the intended output (electricity) direction ($\bar{\theta} = 0$), while the optimal proportionate reduction in coal is positive ($\bar{\delta}_z > 0$) and is such that the new level of coal ($\bar{x}_z - \bar{\delta}_z \bar{x}_z$) lies in the isoquant of $\bar{y}$ level of electricity. However, for these plants, although the efficiency improvement in the coal direction is positive ($\bar{\delta}_z > 0$), it tends to be very small. This could be because the output-based productive efficiency $\mathbf{b}_y$ of these plants is very high, so that they are already very close to the weakly efficient frontier of sub-technology $T_1$.

[50] Recall that the output-based efficiency improvement in the intended output direction is given by $\bar{\beta}_y - 1$, where $\bar{\beta}_y$ is the inverse of $\mathbf{b}_y$.

[51] See also Table 2, where the descriptive statistics of $1 - \bar{\beta}_z$ are provided.

**Table 3: Results with $w_y = w_z = 0.5$**

| | | Performance categories | | |
|---|---|---|---|---|
| | | high | moderate | low |
| **Solution categories*** | (4,2) | 4 | 0 | 0 |
| | (2,1) | 9 | 17 | 0 |
| | (1,1) | 1 | 2 | 2 |
| | (3,2) | 0 | 1 | 10 |
| | (3,3) | 0 | 0 | 1 |
| **θ** | Max | 0.007 | 0.256 | 2.712 |
| | Min | 0 | 0 | 0.247 |
| | Avg | 0 | 0.018 | 0.716 |
| **$\delta_z$** | Max | 0.072 | 0.321 | 0.242 |
| | Min | 0 | 0 | 0 |
| | Avg | 0.024 | 0.184 | 0.022 |
| **$100*(\gamma-(1-\beta_z))/\gamma$** | Max | 37.277 | 84.473 | 68.144 |
| | Min | 0 | 0 | 0 |
| | Avg | 15.190 | 56.162 | 7.893 |
| **$\gamma-(1-\beta_z)$** | Max | 0.064 | 0.292 | 0.218 |
| | Min | 0 | 0 | 0 |
| | Avg | 0.022 | 0.165 | 0.019 |

*Refer to Table 1 for all categories of possible solutions and their interpretations

In fact, these plants must be close to the efficient frontier of $T_1$. Hence, there is very limited scope for efficiency improvements in both electricity and coal directions. It follows that the small proportional reductions in the usage of the coal input at the optima of these plants will induce only small proportional reductions in their $CO_2$ emissions. For these plants, Table 3 shows that the optimal proportional reduction in $CO_2$ emission induced purely due to reductions in the emission-causing input (given by $\bar{\gamma} - (1 - \bar{\beta}_z)$) varies between 0.2% and 6%.[52]

To sum up, under a weighting scheme that gives equal and exhaustive weights to the good and bad outputs, for almost all plants with high output-based productive efficiency, graph efficiency improvement does not imply significant efficiency improvement in the environmental direction and implies no efficiency improvement in the intended output direction. The scope for proportional reduction in $CO_2$ emission, purely due to proportional reduction in usage of coal, is limited for these plants. The analysis demonstrates that, for these plants, there is very little scope for improvement in TDE and reduction in EPUE as defined in (19) because of the close proximity of their production plans to the efficient frontier of sub-technology $T_1$ and to the lower frontier of sub-technology $T_2$.

[52]See (18) for decomposition of $\bar{\gamma}$ into (i) the part attributed to no change in the usage of the emission-causing input $(1 - \bar{\beta}_z)$ and (ii) the part attributed to reduction in this input $\bar{\gamma} - (1 - \bar{\beta}_z)$.

### 9.1.2 Optimal efficiency improvements in the case of the moderate performers.

Table 3 shows that the optimum for a majority of moderate performing plants (17 out of 20 plants) is also positioned in Row **(2)** and Column **(1)** of Table 1. Hence, like in the case of the majority of high performers studied in Section 9.1.1, the optimum recommends that these plants continue producing their respective initial levels of electricity $\bar{y}$ with maximum feasible reduction in usage of the coal input, *i.e.,* they hit the isoquants corresponding to their respective initial levels of electricity generation.

However, since the the output-based efficiency index $\mathbf{b}_y$ takes a lower value for these plants as compared to the high performers, these plants are placed farther below the weakly efficient frontier of sub-technology $T_1$ than the high performers. Hence, unlike in the case of the majority of the high performers, the optimum recommends significant reduction in coal usage for the majority of the moderate performers. Table 3 shows that, in the category of moderate performers, the proportional reduction in coal can be as high as 32% with the average value of $\bar{\delta}_z$ being 18%. This implies that the efficiency improvement in the emission direction induced purely due to reduction in coal is very high for these plants: Table 3 shows that $\bar{\gamma} - (1 - \bar{\beta}_z)$ can be as high as 29% with an average of 16.5% in this category. A large part of the total proportional reduction in emission $\bar{\gamma}$ is attributed to reduction in coal: Table 3 shows that, for the moderate performers, the share of $\bar{\gamma} - (1 - \bar{\beta}_z)$ in $\bar{\gamma}$ can be as high as 84% (this is observed in the case of the thermal power plant Korba West), while the average share is 56%.

To sum up, under a weighting scheme that gives equal and exhaustive weights to the good and bad outputs, the computation of the weighted index of graph efficiency improvements $\mathcal{I}^G$ implies that efficiency improvement in the environmental direction is maximum for the moderate performers (plants with moderate output based productive efficiency index $\mathbf{b}_y$). These turn out to be the plants with the maximum potential for reduction in coal usage, and hence also the plants with the greatest potential for reducing of $CO_2$ emission in our dataset under this weighting scheme. From this and (19) we can also conclude that improvements in TDE and reductions in EPUE at the optima of these plants are brought about mainly due to the reductions in usage of coal (and hence in generation of emission) while producing their existing levels of electricity.

### 9.1.3 Optimal efficiency improvements in the case of the low performers.

These plants have the lowest values of $\mathbf{b}_y$, the output-based productive efficiency index, in our dataset. This implies that they are the farthest away from the weakly efficient frontier of sub-technology $T_1$, so that the potential for efficiency improvement in the electricity direction when all inputs are held fixed (given by $\bar{\beta}_y - 1$, where $\bar{\beta}_y$ is the inverse of $\mathbf{b}_y$) is the highest for these plants among all plants in out dataset. It turns out that, for most of these plants, the optimum of problem (2) recommends that they focus primarily on improving efficiency along the intended output direction.

This is indicated in Table 3, which shows that, for a majority of these plants (11 out of the 13 in this category), the optimum of problem (2) is positioned in Row **(3)** of Table 1. As seen in Section 6, this implies that the optimal proportional increase in electricity for these plants ($\bar{\theta}$) is given by $\bar{\beta}_y - 1$, which as argued above, is the highest for these plants among all plants in our dataset. Table 3 shows that the average value of $\bar{\theta}$ in this category is 72%. The maximum value that $\bar{\theta}$ takes in this category is 271%, and this is observed in the case of thermal power plant Ennore, whose output-based productive efficiency $\mathbf{b}_y$ is 0.269, the lowest in our dataset. Sikka and Rajghat are the thermal power plants in our dataset with the next lowest values of $\mathbf{b}_y$ (0.52 and 0.48, respectively). Hence, for these plants too, the optimal proportional increase in electricity is very high ($\bar{\theta} = \bar{\beta}_y - 1$ takes values 102% and 94%, respectively).

Further, the optima of 10 out of the 11 plants discussed above are positioned in Column **(2)** of Table 1, while the optimum of the eleventh plant (Bhusawal) is positioned in Column **(3)** of Table 1. In either case, we find that the optima of these plants recommend no proportional decreases in the usage of coal input, *i.e.,* $\bar{\delta}_z$ is zero for these plants. This means that the optimal efficiency improvement in the emission direction $\bar{\gamma}$ will be purely output-based, *i.e.,* it will be given by $1 - \bar{\beta}_z$, the maximum possible proportional reduction in $CO_2$ when coal is held fixed at the original level $\bar{x}_z$. For low performers, whose optimum is positioned in Column **(2)** of Table 1, $\bar{\gamma} = 1 - \bar{\beta}_z$ is positive (see Section 6). But as we have already explained, $1 - \bar{\beta}_z$ is small for all plants in our dataset, including the low performers. On the other hand, for Bhusawal, whose optimum is positioned in Column **(3)** of Table 1, $\bar{\gamma} = 1 - \bar{\beta}_z$ is zero, indicating that it lies on the lower frontier of sub-technology $T_2$. (See Section 6)

To sum up, low performers being plants with very low values of output-based productive efficiency $\mathbf{b}_y$, are also the plants that have the greatest potential for increasing electricity generation without changing their existing levels of usage of coal. Under a weighting scheme that gives equal weights to proportional reduction in emission ($\bar{\gamma}$) and proportional increase in electricity ($\bar{\theta}$), the optimum of problem (2) recommends that these plants focus exclusively on tapping this huge potential to increase electricity generation with their existing levels of usage of coal. Hence, (19) implies that, at the optima of these plants, improvements in TDE and reductions in EPUE are attributed mainly to expansion in generation of electricity with their existing usage of the coal input.

### 9.1.4 The plants with interior solutions to problem (2).

Even under a weighting scheme that gives equal weights to proportional reductions in the intended output electricity and the $CO_2$ emission, Table 3 and our discussion in Sections 9.1.1, 9.1.2, and 9.1.3 showed that, for a majority of plants, the optimum of problem (2) either recommends efficiency improvement only along the coal and emission directions ($\bar{\delta}_z > 0$ and $\bar{\gamma} > 0$) with no increase in electricity generation ($\bar{\theta} = 0$) (this is the case of high and moderate performers), or it recommends huge efficiency improvement along the electricity direction ($\bar{\theta} > 0$) with no reduction in coal ($\bar{\delta}_z = 0$) and very limited (output-based) reduction in emission generation (this is the case of low performers). Thus, for a majority of plants in our dataset, problem (2) yields corner solutions even under a scheme with equal weighting of the good and the bad outputs. This means that, for majority of the high and moderate performers, at the optimum of problem (14), the marginal gain from reducing the emission-causing input coal (given by the increase in $\gamma$) is higher than the marginal cost of reducing coal (given by the reduction in $\theta$). For the low performers, the reverse is true: the marginal cost of reduction in usage of coal (given by the reduction in $\theta$) is higher than the marginal gain (given by the increase in $\gamma$).

**Table 4: Results for plants with interior solutions**

| Name | Performance | θ | γ | $\delta_z$ | γ-(1-$\beta_z$) |
|---|---|---|---|---|---|
| Talcher | high | 0.007 | 0.111 | 0.028 | 0.026 |
| Farakka | moderate | 0.011 | 0.182 | 0.090 | 0.081 |
| Kahalgaon | moderate | 0.087 | 0.107 | 0.009 | 0.008 |
| Raichur | low | 0.247 | 0.132 | 0.039 | 0.035 |
| GNDTPS (Bhatinda) | low | 0.326 | 0.319 | 0.242 | 0.218 |

Table 3 shows that there are only five plants in our data set for which problem (2) yields interior solutions when equal and exhaustive weights are assigned to the good and bad outputs (*i.e.*, for these plants $\bar{\delta}_z > 0$, $\bar{\gamma} > 0$, and $\bar{\theta} > 0$). All these plants are positioned in Row **(1)** and Column **(1)** of Table 1. For these plants, the marginal gains and losses from reduction in coal are equalised at the optimum of problem (2) or problem (14). As seen in Table 4, one (namely, Talcher) is a high performer, two (namely, Farakka and Kahalgaon) are moderate performers, and two (namely, Raichur and Bhatinda) are low performers. Among these, the solution for Bhatinda involves the greatest proportional increases in both the electricity and emission direction (both $\bar{\theta}$ and $\bar{\gamma}$ are around 32%). It follows that for these five plants, improvements in TDE and reductions in EPUE at the optimum are attributable to both increase in electricity generation and reduction in coal usage (and hence emission generation).

## 9.2 Effects of varying weights on $\theta$ and $\gamma$.

In this section we study the effects of putting more and more weight on reduction in emission starting from a zero weight. When a zero weight is assigned to emission reduction so that only increases in electricity generation matter to the researcher or policy maker, then it is intuitive that efficiency improvements at the optimum of problem (2) will coincide with output-based efficiency improvements in Murty and Nagpal (2018). Reduction in coal and the induced reduction in emission are not valued in this case. In fact, reduction in coal will imply that the efficiency improvement in electricity generation, which is assigned full weight, can reduce. Hence, the optimum will recommend no reduction in usage of coal.

Table 5 summarises the results when weight on proportionate reduction in emission is increased gradually from zero to one-third, half, two-thirds, and finally to one.

When the weight on emission reduction is chosen to be one-third and a large weight of two-thirds is assigned to efficiency improvement in the electricity direction, then Table 5 shows that for a big majority of plants the optimum continues to recommend maximum proportional increase in electricity with no change in the usage of the coal input. 43 out of the 47 plants in our dataset are positioned in Row **(3)** of Table 1, where the optimal proportional increase in electricity ($\bar{\theta}$) is given by the output-based measure $\bar{\beta}_y - 1$.

**Table 5: Result under different weighting schemes**

| | | $w_y$= 2/3, $w_z$= 1/3 | $w_y$= 1/2, $w_z$= 1/2 | $w_y$= 1/3, $w_z$= 2/3 | $w_y$= 0, $w_z$= 1 |
|---|---|---|---|---|---|
| **Solution categories*** | (4,2) | 4 | 4 | 4 | 4 |
| | (2,1) | 0 | 26 | 38 | 43 |
| | (1,1) | 0 | 5 | 3 | 0 |
| | (3,2) | 39 | 11 | 2 | 0 |
| | (3,3) | 1 | 1 | 0 | 0 |
| | (3,1) | 3 | 0 | 0 | 0 |
| **$\gamma$-(1-$\beta_z$)** | Max | 0.025 | 0.292 | 0.520 | 0.685 |
| | Min | 0 | 0 | 0 | 0 |
| | Avg | 0.001 | 0.082 | 0.161 | 0.195 |

*Refer to Table 1 for all categories of possible solutions and their interpretations

Table 5 also indicates that 39 out of the 43 plants mentioned above are positioned in Column **(2)** of Table 1. Hence, the optimal proportional reduction in emission ($\bar{\gamma}$) for these plants is also given by the output-based measure $1 - \bar{\beta}_z$. At the same time, the optimum recommends no reduction in usage of coal for these plants.

Table 5 shows that only 3 out of the 43 plants mentioned above, namely, thermal plants Kota, Singrauli, and Tuticoran, are positioned in Column **(1)** of Table 1. Hence, for these plants, the optimum recommends reduction in the usage of coal, which implies that efficiency improvement in the emission direction $\bar{\gamma}$ will be greater than the output-based measure $1-\bar{\beta}_z$ for these plants. As the optimum is positioned in Row **(3)** of Table 1 so that $\bar{\theta}$ is equal to the output-based measure $\bar{\beta}_y - 1$, it follows from the discussion in Section 6.2 that there is a slack on the weakly efficient frontier of sub-technology $T_1$, *i.e.* efficiency improvement in the coal direction happens without compromising on the extent of efficiency improvement in the electricity direction.

Results from the case where equal weights are assigned to $\theta$ and $\gamma$ were discussed in detail in Section 9.1. We only note here that, when the weight on emission is increased from one-third to half, the number of plants for which the optimum recommends efficiency improvement in the coal input direction increases from 3 in the previous case to 31.[53] On the other hand, the number of plants for which the optimum recommends (output-based) efficiency improvement in the electricity direction with no change in usage of coal falls from 40 to 12.[54] It is clear that for the 31 plants for which the optimum recommends reduction in coal usage, the proportionate reduction in emission will be in excess of the output-based measure $1-\bar{\beta}_z$.

When the weight on emission reduction is increased further to two-third, Table 5 shows that the number of the plants for which the optimum recommends proportionate reduction in usage of coal increases to 41.[55] In all these plants, proportionate reduction in emission is induced beyond the output-based measure $1-\bar{\beta}_z$.

Table 5 shows that, when full weight is assigned to emission reduction, then for all but the four productively efficient plants[56] the optimum recommends maximum feasible proportionate decrease in coal usage with no efficiency improvement in the electricity direction, *i.e.,* 43 of the 47 plants in our data set are positioned in Row **(2)** and Column **(1)** of Table 1. Thus, under this weighting scheme, the optimum of problem (2) recommends emission reduction in excess of the output-based measure $1-\bar{\beta}_z$ for all these 43 plants.

Thus, as the weight assigned to proportionate reduction in emission increases, gains in TDE and reductions in EPUE at the optimum of problem (2) are increasingly attributed to reduction in coal usage (and hence emission reduction) and not so much to expansion in electricity generation.

Table 5 also shows that as more and more weight is assigned to proportionate reduction in emission, the maximum and the average proportionate increase in emission induced by reduction in coal usage $\bar{\gamma}-(1-\bar{\beta}_z)$ increase. The maximum increases from 0.2% when weight on emission

[53]Table 5 shows that 26 of the 31 plants are positioned in Row **(2)** and Column **(1)** of Table 1 and 5 of these plants positioned in Row **(1)** and Column **(1)** of this table.

[54]Table 5 shows that these are plants with positions $(3,2)$ or $(3,3)$ in Table 1.

[55]The plants positioned in Row **(2)** and Column **(1)** of Table 1 increases to 39, while there are 2 positioned in Row **(1)** and Column **(1)** of this table.

[56]Namely, Bokaro B, Dahanu, Vindhyanchal and Korba (positioned in Row **(4)** and Column **(2)** of Table 1)

is one-third to 69% when full weight is assigned to it.

**Table 6: List of plants with maximum potential to reduce emission under different weighting schemes**

| $w_y$= 2/3, $w_z$ =1/3 | | | $w_y$= 1/2, $w_z$ =1/2 | | |
|---|---|---|---|---|---|
| **Name** | **Performance** | **$\gamma$-(1-$\beta_z$)** | **Name** | **Performance** | **$\gamma$-(1-$\beta_z$)** |
| Kota | moderate | 0.025 | Korba West | moderate | 0.292 |
| Tuticorin | moderate | 0.024 | Khaperkheda | moderate | 0.287 |
| Singrauli | high | 0.008 | Sanjay Gandhi | moderate | 0.268 |
| | | | Metur | moderate | 0.259 |
| | | | Chandrapur (DVC) | moderate | 0.247 |
| **$w_y$= 1/3, $w_z$ =2/3** | | | **$w_y$= 0, $w_z$ =1** | | |
| **Name** | **Performance** | **$\gamma$-(1-$\beta_z$)** | **Name** | **Performance** | **$\gamma$-(1-$\beta_z$)** |
| Satpura | low | 0.520 | Ennore | low | 0.685 |
| Bhusawal | low | 0.495 | Satpura | low | 0.553 |
| GNDTPS(Bhatinda) | low | 0.385 | Bhusawal | low | 0.495 |
| Parli | low | 0.381 | Rajghat | low | 0.495 |
| Gandhi Nagar | low | 0.370 | Sikka | low | 0.438 |
| | | | Koradi | low | 0.425 |

Table 6 lists the plants with greatest potential to reduce emission due to reductions in the usage of coal under different weighting schemes. It demonstrates the changing nature of these plants as weight on emission reduction is increased. When weight on emission reduction is less than or equal to one-half, plants with highest values of $\bar{\gamma} - (1 - \bar{\beta}_z)$ are moderate and high performers; while when the weight on emission reduction is greater than one-half, plants with maximum potential to reduce emission are the low performers.

# 10 Conclusions.

This study uses the by-production specification of the technology to measure the scope of efficiency improvements for 47 Indian thermal power plants in 2014 by developing a weighted graph index of efficiency improvements. The current study is an extension of a previous study by Murty and Nagpal (2018) where output-based efficiency index is used for measuring efficiency of Indian thermal power plants. Given the way $CO_2$ data is computed by the Central Electricity Authority of India, measurement of environment efficiency under output-based efficiency index does not give meaningful results. Moreover, it is not possible to gauge the maximum potential of a power plant for reducing $CO_2$ emission when inputs, especially coal, are held fixed. Graph efficiency index overcomes these inadequacies by allowing all inputs and outputs to vary.

We compare the empirical results of weighted graph efficiency index when equal weights are assigned to electricity-expansion and emission-reduction with the results obtained under purely output-based FGL measure of technical efficiency. We find that the optimal configuration of efficiency improvement varies depending upon a plant's proximity to the frontier (as measured by the output-based FGL index). It is optimal for most high performing power plants (plants closest to the frontier) to reduce their usage of coal keeping electricity unaltered. However, they have the least scope of efficiency improvement in either the intended output direction or the environment direction. The optimal configuration of efficiency improvements for a majority of moderate performing plants also involves reduction in coal keeping electricity generation fixed, while the optimal configuration for low performing plants (plants farthest from the frontier) suggests expansion in electricity generation keeping coal usage unchanged. The moderate performing plants exhibit maximum potential for environment efficiency improvement while the low performing plants have the maximum scope for increasing electricity generation.

However, the optimum for a power plant changes with the change in the weighing scheme. The plants that have maximum potential for expansion of the intended output when more weight is assigned to electricity generation become plants with maximum potential for $CO_2$ reduction when more weight is assigned to emission reduction. Thus, in the presence of technical inefficiencies, the policymakers need to choose an appropriate system of weights while designing and implementing policies in accordance with the contrasting needs of a nation: increasing electricity generation and/or reducing emission. In particular, weighted graph efficiency measures are helpful in understanding the potential for emission reduction when the objective is to meet a lower emission target without compromising on electricity generation. A scheme in place for Indian thermal power plants called Perform, Achieve, and Trade (PAT) promotes improvement in thermodynamic efficiency through lower station heat rate, but it does not rule out an increase in overall emissions as electricity generation increases.[57] In contrast, the efficiency improvements for an inefficient plant derived through measures such as the weighted graph index of efficiency improvements developed in this paper would ensure improvement in thermodynamic efficiency and reduction in emission per unit of electricity along with adherence to an overall emission cap. As countries commit to lowering their $CO_2$ emission levels, while at the same time strive towards expanding their respective thermal power sectors, the need for developing methodologies that compute optimal configurations of efficiency improvements gain significance. In this backdrop, this study is useful in understanding how the plant-level optimum changes as priorities and preferences of policymakers for reduction in emission and expansion of desirable output shift.

## REFERENCES

Baumgärtner, S., and J. de Swaan Arons (2003), "Necessity and Inefficiency in the Generation

[57]The station heat rate is the inverse of thermodynamic efficiency. Hence, it measures the heat input used per unit of gross electricity generation.

of Waste," *Journal of Industrial Ecology*, 7: 113–123.

Baumol, W. J., and W. E. Oates (1975, 1988), *The Theory of Environmental Policy*, 1st and 2nd editions, Cambridge University Press.

Behera, S.K., A.P. Dash and J. A. Farooquie (2010), "Performance Analysis of Coal Fired Power Plants in India," *Proceedings of the 2010 International Conference on Industrial Engineering and Operations Management Dhaka*, Bangladesh.

Coal Industry Advisory Board (CIAB) (2010) "Power Generation from Coal Measuring and Reporting Efficiency Performance and CO2 Emissions," *International Energy Agency (IEA)*

Coggins, J. S., and J. R. Swinton (1996), "The Price of Pollution: A Dual Approach to Valuing $SO_2$ Allowances," *Journal of Environmental Economics and Management*, 30: 58–72.

Cropper, M. L., and W. E. Oates (1992), "Environmental Economics: A Survey," *Journal of Economic Literature*, 30: 675–740.

Dapko, K. H. (2015), "On Modeling Pollution-Generating Technologies: A New Formulation of the By-production Approach," Paper presented at the 6th EAAE PhD Workshop, Rome Co-organized by AIEAA (Italian Association of Agricultural and Applied Economics) and the Department of Economics of Roma Tre University.
$http://prodinra.inra.fr/ft?id = 1DE1A17F - F41C - 41F9 - A853 - 305563DA93A9$

Debreu, G. (1959), *Theory of value*, New York, Wiley

Färe, R., S. Grosskopf, C. A. K. Lovell, and C. Pasurka (1989), "Multilateral Productivity Comparisons When Some Outputs are Undesirable: A Nonparametric Approach," *The Review of Economics and Statistics*, 71: 90–98.

Färe, R., and C.A.K. Lovell (1978) "Measuring the technical efficiency of production," *Journal of Economic Theory* 19, 150–162.

Färe, Rolf, S. Grosskopf, D. W. Noh, and W. Weber (2005). "Characteristics of a Polluting Technology: Theory and Practice." *Journal of Econometrics*, 126: 469–492.

Førsund, F. (1998), "Pollution Modeling and Multiple-Output Production Theory," Discussion Paper #D-37/1998, Department of Economics and Social Sciences, Agricultural University of Norway.

Førsund, F. (2009), "Good Modelling of Bad Outputs: Pollution and Multiple-Output Production," *International Review of Environmental and Resource Economics*, 3, 1–38.

Førsund, F. (2017), "Multi-equation Modeling of Desirable and Undesirable Outputs Satisfying the Material Balance," *Empirical Economics*, online.

Hailu, A., and T. S. Veeman (2000), "Environmentally Sensitive Productive Analysis of the Canadian Pulp and Paper Industry, 1959–1994: an input distance function approach," *Journal of Environmental Economics and Management*, 40, 251–274

Kumar, R., A. K. Sharma, and P. C. Tewari (2015), "Cost analysis of a coal-fired power plant using the NPV method," *Journal of Industrial Engineering International*, DOI 10.1007/s40092-015-0116-8

Kumbhakar, S. C., and E. G Tsionas (2016), "The Good, the Bad and the Technology: Endogeneity in Environmental Production Models", *Journal of Econometrics*, in press, 1–13.

Lee, J. D., J. B. Park, T. Y. Kim (2002), "Estimation of the shadow prices of pollutants with production/environment inefficiency taken into account: A nonparametric directional distance function approach," *Journal of Environmental Management*, 63, 365-375

Malikov, E., Kumbhakar, C., and Tsionas, E. G. 2015. Bayesian Approach to Disentangling Technical and Environmental Productivity. *Econometrics*, 3: 443–465.

Mas-Colell, A., M. D. Whinston, and J. R. Green (1995) *Microeconomic theory*, Chp. 5. Oxford University Press, New York

Ministry of Power (2010), Government of India, Ministry of Power, Central Electricity Authority, "Mapping of 85 pulverized coal fired thermal power generating units in different states," under Indo-German Energy Program (IGEN).

Ministry of Power (2013-14, 2014-15), *Review of Performance of Thermal Power Plants in India*, Central Electricity Authority, Ministry of Power, Government of India.

Murty, M. N., and S. Kumar (2002), "Measuring Cost of Environmentally Sustainable Industrial Development in India: A Distance Function Approach," *Environment and Development Economics*, 7: 467–86.

Murty, M. N., and S. Kumar (2003), "Win-Win Opportunities and Environmental Regulation: Testing of Porter Hypothesis for Indian Manufacturing Industries," *Journal of Environmental Management*, 67: 139–44.

Murty, S (2015), "On the Properties of an Emission-Generating Technology and its Parametric Representation," *Economic Theory*. 60, 243–282.

Murty, S, and R. Nagpal (2017), "Measuring Output-Based Technical Efficiency of Indian Coal-Based Thermal Power Plants: A By-production Approach," Discussion Paper 18-07, Centre for International Trade and Development, School of International Studies, Jawaharlal Nehru University. *https://jnu.ac.in/sites/default/files/citd/CITD-DP-Murty%26NagpalOuputBasedEfficiency2018.pdf*

Murty, S, and R. R. Russell (2002), "On Modeling Pollution-Generating Technologies," Department of Economics, University of California, Riverside, Discussion Papers Series, No. 02-14.

Murty, S., R. R. Russell, and S. B. Levkoff (2012), "On Modeling Pollution-Generating Technologies," *Journal of Environmental Economics and Management*, 64, 117–135.

Murty, S., and R. R. Russell (2016), "Modeling Emission-Generating Technologies: Reconciliation of Axiomatic and By-Production Approaches," *Empirical Economics*, online: DOI

10.1007/s00181-016-1183-4.

Murty, S., and R. R. Russell (2017) "Bad outputs," *Discussion Papers in Economics* No. 17-06, Centre for International Trade and Development, School of International Studies, Jawaharlal Nehru University, India.

Pasurka, C. A. (2006) "Decomposing electric power plant emissions within a joint production framework," *Energy Economics* 28 26–43.

Ray, S. C., K. Mukherjee, and A. Venkatesh (2017), "Nonparametric Measures of Efficiency in the Presence of Undesirable Outputs: A By-production Approach with Weak Disposability," *Empirical Economics*.

Reinhard, S, C. A. K. Lovell, and G. J. Thijssen, (1999), "Econometric Estimation of Technical and Environmental Efficiency: An Application to Dutch Dairy Farms," *American Journal of Agricultural Economics*, 81: 44–60.

Reinhard, Stijn, C. A. Knox Lovell, and Geert J. Thijssen (2000) "Environmental Efficiency with Multiple Environmentally Detrimental Variables; Estimated with SFA and DEA." *European Journal of Operational Research*, 121: 287–303.

Russell, R. R. (1998) *Distance functions in consumer and producer theory* in Färe R., Grosskopf, S., Russell R.R. (eds.) Index numbers: Essays in honor of Sten Malmquist. Kluwer, Boston

Sahoo, N. R., Pratap K.J. Mohapatra, B. K. Sahoo, and B. Mahanty (2017) "Rationality of Energy Efficiency Improvement Targets under the PAT Scheme in India – A Case of Thermal Power Plants," *Energy Economics* doi:10.1016/j.eneco.2017.06.004

Serra, T., R. G. Chambers, and A. O. Lansink (2016), "Measuring Technical and Environmental Efficiency in a State-Contingent Techology," *European Journal of Operational Research*, 236: 706–717.

Shephard, R. W. (1953) *Cost and production functions*, Princeton University Press, Princeton

Sueyoshi, T., and M. Goto (2010), "Should the US Clean Air Act Include $CO_2$ Emission Control?: Examination by Data Envelopment Analysis," *Energy Policy*, 38, 5902–5911

Sueyoshi, T., and M. Goto (2011), "DEA approach for unified efficiency measurement: Assessment of Japanese fossil fuel power generation," *Energy Economics*, 33, 292–303

Sueyoshi, T., and M. Goto (2012), "DEA environmental assessment of coal fired power plants: Methodological comparison between radial and non-radial models," *EnergyEconomics*, 34, 1854–1863

Zhou P., B. Ang, and K-L Poh (2008), "A Survey of Data Envelopment Analysis in Energy and Environmental Studies," *European Journal of Operational Research*, 189: 1–18.

# APPENDIX

**Proof. (Theorem 2)**

(i) Proving monotonicity of $\Theta$ and $\Gamma$:
Suppose $\langle \bar{\delta}_o, \bar{\delta}_z \rangle$ and $\langle \delta'_o, \delta'_z \rangle$ are both in $[0,1]^n$ and $\langle \bar{\delta}_o, \bar{\delta}_z \rangle \geq \langle \delta'_o, \delta'_z \rangle$.
Let $\theta' = \Theta\left(\delta'_o, \delta'_z\right)$ and $\bar{\theta} = \Theta\left(\bar{\delta}_o, \bar{\delta}_z\right)$. Then

$$\left\langle \bar{x}_o - \left(\bar{\delta}_o \otimes \bar{x}_o\right),\ \bar{x}_z - \left(\bar{\delta}_z \otimes \bar{x}_z\right),\ \bar{y} + \bar{\theta}\bar{y},\ \bar{z} \right\rangle \in T_1.$$

Since $T_1$ satisfies free input disposability and

$$\left\langle \bar{x}_o - \left(\delta'_o \otimes \bar{x}_o\right),\ \bar{x}_z - \left(\delta'_z \otimes \bar{x}_z\right) \right\rangle \geq \left\langle \bar{x}_o - \left(\bar{\delta}_o \otimes \bar{x}_o\right),\ \bar{x}_z - \left(\bar{\delta}_z \otimes \bar{x}_z\right) \right\rangle,$$

we have

$$\left\langle \bar{x}_o - \left(\delta'_o \otimes \bar{x}_o\right),\ \bar{x}_z - \left(\delta'_z \otimes \bar{x}_z\right),\ \bar{y} + \bar{\theta}\bar{y},\ \bar{z} \right\rangle \in T_1.$$

Therefore, $\bar{\theta}$ is in the constraint set of problem (8) when $\langle \delta_o, \delta_z \rangle = \langle \delta'_o, \delta'_z \rangle$. Hence $\theta' \geq \bar{\theta}$.

From condition (ii') of Definition 2 of a BPT it follows that the lower frontiers of sets $T_2$ and $CDH\left(T_2\right)$ are the same. Hence, the definition of function $\Gamma$ in (9) also satisfies:

$$\Gamma\left(\delta_z\right) \quad = \quad \max_{\gamma \in \mathbf{R}_+} \left\{ \gamma \;\middle|\; \left\langle \bar{x}_o,\ \bar{x}_z - \left(\delta_z \otimes \bar{x}_z\right),\ \bar{y},\ \bar{z} - \gamma\bar{z} \right\rangle \in CDH\left(T_2\right) \right\}.$$

Suppose $\bar{\delta}_z \geq \delta'_z$ and let $\gamma' = \Gamma\left(\delta'_z\right)$ and $\bar{\gamma} = \Gamma\left(\bar{\delta}_z\right)$. From the definition of $\Gamma$, this implies:

$$\left\langle \bar{x}_o,\ \bar{x}_z - \left(\delta'_z \otimes \bar{x}_z\right),\ \bar{y},\ \bar{z} - \gamma'\bar{z} \right\rangle \in CDH\left(T_2\right).$$

Since $CDH\left(T_2\right)$ satisfies the assumptions of costly disposability emission and emission-causing inputs, and

$$\bar{x}_z - \left(\delta'_z \otimes \bar{x}_z\right) \geq \bar{x}_z - \left(\bar{\delta}_z \otimes \bar{x}_z\right),$$

we have

$$\left\langle \bar{x}_o,\ \bar{x}_z - \left(\bar{\delta}_z \otimes \bar{x}_z\right),\ \bar{y},\ \bar{z} - \gamma'\bar{z} \right\rangle \in CDH\left(T_2\right).$$

Therefore $\gamma'$ is in the constraint set of problem (9) when $\delta_z = \bar{\delta}_z$. Hence $\bar{\gamma} \geq \gamma'$.

(ii) Proving concavity of $\Theta$ and $\Gamma$:

We need to show that the hypographs of functions $\Theta$ and $\Gamma$ are convex sets.[58] Let $\delta := \langle \delta_o, \delta_z \rangle$ and $\hat{\delta} := \langle \hat{\delta}_o, \hat{\delta}_z \rangle$ lie in $\Omega$ and define $\delta' = \alpha\delta + (1-\alpha)\hat{\delta}$, where $\alpha \in [0,1]$. Let $\Theta(\delta) = \theta$, $\Theta\left(\hat{\delta}\right) = \hat{\theta}$, and $\Theta(\delta') = \theta'$. Thus, $\langle \delta, \theta \rangle$, $\langle \hat{\delta}, \hat{\theta} \rangle$ lie in the hypograph of $\Theta$. To show that the hypograph of $\Theta$ is a convex set, we need to show that $\langle \delta', \theta^* \rangle$ is also in the hypograph of $\Theta$, where $\theta^* = \alpha\theta + (1-\alpha)\theta'$. That is, we need to show that

$$\theta^* \leq \Theta(\delta') \equiv \theta'.$$

The definition of function $\Theta$ implies that $v = \langle \bar{x} + \delta \otimes \bar{x},\ \bar{y} + \theta\bar{y},\ \bar{z} \rangle \in T_1$, $\hat{v} = \langle \bar{x} + \hat{\delta} \otimes \bar{x},\ \bar{y} + \hat{\theta}\bar{y},\ \bar{z} \rangle \in T_1$, and $v' = \langle \bar{x} + \delta'\bar{x},\ \bar{y} + \theta'\bar{y},\ \bar{z} \rangle \in T_1$. Since $T_1$ is a convex set, we have $\alpha v + (1-\alpha)\hat{v} \in T_1$. But $\alpha v + (1-\alpha)\hat{v} = \langle \bar{x} + \delta' \otimes \bar{x},\ \bar{y} + \theta^*\bar{y},\ \bar{z} \rangle \in T_1$. Thus, $\theta^*$ is in the constraint set of problem (8) when the proportional change in inputs is given by the vector $\delta'$. Hence, the definition of $\Theta$ implies that $\theta^* \leq \Theta(\delta')$. Hence, $\langle \delta', \theta^* \rangle$ is in the hypograph of $\Theta$.

The proof of concavity of function $\Gamma$ proceeds in an exactly similar manner. ■

**Proof. (Image of function $\Theta$)** If $\langle \delta_o, \delta_z \rangle = 0_n$ then $\langle \bar{x}_o - (\delta_o \otimes \bar{x}_o),\ \bar{x}_z - (\delta_z \otimes \bar{x}_z), \bar{y}, \bar{z} \rangle = \langle \bar{x}_o, \bar{x}_z, \bar{y}, \bar{z} \rangle \in T_1$. Hence, $\langle \delta_o, \delta_z \rangle = 0_n \in L_\delta$. Thus, the constraint set of problem (8) when $\langle \delta_o, \delta_z \rangle = 0_n$ becomes $\{\theta \in \mathbf{R} \mid \langle \bar{x}_o, \bar{x}_z, (1+\theta)\bar{y}, \bar{z} \rangle \in T_1\}$. Comparison with solution to problem (5) implies that $(1+\bar{\theta})\bar{y} = \bar{\beta}_y\bar{y}$, which implies $\bar{\theta} = \Theta(\delta_o, \delta_z) = \bar{\beta}_y - 1$.

The definitions of functions $\Theta$ and $F$ imply that if $\langle \delta_o, \delta_z \rangle \in I_\delta$ then $\Theta(\delta_o, \delta_z) = 0$.

Suppose $\langle \delta_o, \delta_z \rangle \in \Omega \setminus L_\delta$ and $\langle \bar{x}_o - (\delta_o \otimes \bar{x}_o),\ \bar{x}_z - (\delta_z \otimes \bar{x}_z) \rangle \in L(y)$ with $y \geq \bar{y}$. Free output disposability of $T_1$ implies that $L(y) \subseteq L(\bar{y})$. Hence, $\langle \bar{x}_o - (\delta_o \otimes \bar{x}_o),\ \bar{x}_z - (\delta_z \otimes \bar{x}_z) \rangle \in L(\bar{y})$, which is a contradiction to $\langle \delta_o, \delta_z \rangle \in \Omega \setminus L_\delta$. Hence, $\langle \delta_o, \delta_z \rangle \in \Omega \setminus L_\delta$ implies $\langle \bar{x}_o - (\delta_o \otimes \bar{x}_o),\ \bar{x}_z - (\delta_z \otimes \bar{x}_z) \rangle \in L(y)$ with $y < \bar{y}$. This implies, $\langle \bar{x}_o - (\delta_o \otimes \bar{x}_o),\ \bar{x}_z - (\delta_z \otimes \bar{x}_z), y, z \rangle \in T_1$ for all $z \geq 0$. Hence, if $\theta$ is in the constraint set of problem (8), we have $y = \bar{y} + \theta\bar{y}$. Since $y < \bar{y}$, this is true when $\theta < 0$. Hence, in this case, $\Theta(\delta_o, \delta_z) < 0$. Non-negativity of the intended output implies that there is lower bound on the value function $\Theta$ can take: $\Theta(\delta_o, \delta_z) \geq -1$. Thus, for $\delta \in \Omega \setminus L_\delta$, we have $-1 \leq \Theta(\delta) < 0$. ■

[58] The hypograph of a function $f : \mathbf{R}^n \longrightarrow \mathbf{R}$ with image $y = f(x)$ is defined as the set $\{\langle x, y \rangle \in \mathbf{R}^{n+1} \mid y \leq f(x)\}$.

**Proof. (Image of function $\Gamma$)** Since set $CDH\left(T_2\right)$ satisfies costly disposability, $\bar{\beta}_z \bar{z}$ level of emission is feasible under set $CDH\left(T_2\right)$ for all emission-causing input vectors smaller that $\bar{x}_z$. Hence, $\gamma = 1 - \bar{\beta}_z$ is a member of the constraint set of problem (13) for all $\delta_z \in \mathcal{Q}_z = [0,1]^{n_z}$. Hence $\Gamma\left(\delta_z\right)$, which is a measure of the maximum proportional reduction in emission under sub-technology $T_2$ or its costly disposal hull $CDH\left(T_2\right)$, cannot take values less than $1-\bar{\beta}_z$. Thus, we conclude that $\Gamma\left(\delta_z\right) \in [1-\bar{\beta}_z, 1]$ whenever $\delta_z \in \mathcal{Q}_z$. In particular, if $\delta_z = 0_{n_z}$, then $\Gamma\left(\delta_z\right) = 1-\bar{\beta}_z$. This is because, in that case, $\langle \bar{x}_o,\ \bar{x}_z - \left(\delta_z \otimes \bar{x}_z\right), \bar{y}, \bar{z}\rangle = \langle \bar{x}_o, \bar{x}_z, \bar{y}, \bar{z}\rangle \in T_2$. Thus, the constraint set of problem (9) when $\delta_z = 0_n$ becomes $\{\gamma \in \mathbf{R}_+ \mid \langle \bar{x}_o, \bar{x}_z, \bar{y}, (1-\gamma)\bar{z}\rangle \in T_2\}$. Comparison with solution to problem (6) implies that $(1-\bar{\gamma})\bar{z} = \bar{\beta}_z \bar{z}$, which implies $\bar{\gamma} = \Gamma\left(\delta_z\right) = 1 - \bar{\beta}_z$. ■

| | Output-based FGL index | | | Graph Efficiency Index | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | $w_z$=1/3,$w_y$=2/3 | | | | | $w_y$=$w_z$=1/2 | | | | |
| **Plant name** | **$b_y$** | **$\beta_y$-1** | **1-$\beta_z$** | **$\theta$** | **$\gamma$** | **Row** | **Col*** | **$\delta_z$** | **$\theta$** | **$\gamma$** | **Row** | **Col*** | **$\delta_z$** |
| Bokaro B | 1 | 0 | 0.160 | 0 | 0.11 | 4 | 2 | 0 | 0 | 0.16 | 4 | 2 | 0 |
| Dahanu | 1 | 0 | 0.106 | 0 | 0.1 | 4 | 2 | 0 | 0 | 0.11 | 4 | 2 | 0 |
| Korba | 1 | 0 | 0.102 | 0 | 0.16 | 4 | 2 | 0 | 0 | 0.1 | 4 | 2 | 0 |
| Vindhyanchal | 1 | 0 | 0.100 | 0 | 0.1 | 4 | 2 | 0 | 0 | 0.1 | 4 | 2 | 0 |
| Singrauli | 0.999 | 0.001 | 0.106 | 0 | 0.11 | 3 | 1 | 0.01 | 0 | 0.11 | 2 | 1 | 0.01 |
| Ramagundem | 0.998 | 0.002 | 0.105 | 0 | 0.1 | 3 | 2 | 0 | 0 | 0.11 | 2 | 1 | 0 |
| Sipat | 0.991 | 0.009 | 0.105 | 0.01 | 0.11 | 3 | 2 | 0 | 0 | 0.11 | 2 | 1 | 0.01 |
| Simhadri | 0.983 | 0.017 | 0.096 | 0.02 | 0.1 | 3 | 2 | 0 | 0 | 0.12 | 2 | 1 | 0.03 |
| Rihand | 0.978 | 0.023 | 0.109 | 0.02 | 0.11 | 3 | 2 | 0 | 0 | 0.14 | 2 | 1 | 0.03 |
| Talcher | 0.972 | 0.029 | 0.085 | 0.03 | 0.08 | 3 | 2 | 0 | 0.01 | 0.11 | 1 | 1 | 0.03 |
| Budge Budge | 0.970 | 0.030 | 0.102 | 0.03 | 0.1 | 3 | 2 | 0 | 0 | 0.14 | 2 | 1 | 0.04 |
| Dadri (NCTPP) | 0.964 | 0.037 | 0.102 | 0.04 | 0.1 | 3 | 2 | 0 | 0 | 0.15 | 2 | 1 | 0.05 |
| Dr. N. Tata Rao | 0.955 | 0.047 | 0.103 | 0.05 | 0.1 | 3 | 2 | 0 | 0 | 0.16 | 2 | 1 | 0.07 |
| Unchahar | 0.951 | 0.052 | 0.108 | 0.05 | 0.11 | 3 | 2 | 0 | 0 | 0.17 | 2 | 1 | 0.07 |
| Kakatiya | 0.931 | 0.075 | 0.100 | 0.07 | 0.1 | 3 | 2 | 0 | 0 | 0.2 | 2 | 1 | 0.11 |
| Tuticorin | 0.927 | 0.079 | 0.095 | 0.08 | 0.12 | 3 | 1 | 0.03 | 0 | 0.21 | 2 | 1 | 0.13 |
| Farakka | 0.925 | 0.081 | 0.101 | 0.08 | 0.1 | 3 | 2 | 0 | 0.01 | 0.18 | 1 | 1 | 0.09 |
| Rayalseema | 0.924 | 0.083 | 0.107 | 0.08 | 0.11 | 3 | 2 | 0 | 0 | 0.21 | 2 | 1 | 0.12 |
| Kota | 0.923 | 0.084 | 0.105 | 0.08 | 0.13 | 3 | 1 | 0.03 | 0 | 0.23 | 2 | 1 | 0.14 |
| Kahalgaon | 0.914 | 0.094 | 0.099 | 0.09 | 0.1 | 3 | 2 | 0 | 0.09 | 0.11 | 1 | 1 | 0.01 |
| Kothagundem | 0.894 | 0.119 | 0.158 | 0.12 | 0.16 | 3 | 2 | 0 | 0 | 0.29 | 2 | 1 | 0.16 |
| IB Valley | 0.889 | 0.125 | 0.095 | 0.12 | 0.09 | 3 | 2 | 0 | 0 | 0.24 | 2 | 1 | 0.17 |
| Chhabra | 0.873 | 0.145 | 0.181 | 0.15 | 0.18 | 3 | 2 | 0 | 0 | 0.34 | 2 | 1 | 0.19 |
| North Chennai | 0.870 | 0.149 | 0.084 | 0.15 | 0.08 | 3 | 2 | 0 | 0 | 0.26 | 2 | 1 | 0.19 |
| Ropar | 0.857 | 0.166 | 0.094 | 0.17 | 0.09 | 3 | 2 | 0 | 0 | 0.29 | 2 | 1 | 0.21 |
| GHTPS (Lehra Mohabbat) | 0.850 | 0.177 | 0.105 | 0.18 | 0.11 | 3 | 2 | 0 | 0 | 0.3 | 2 | 1 | 0.22 |
| Nasik | 0.848 | 0.179 | 0.097 | 0.18 | 0.1 | 3 | 2 | 0 | 0 | 0.3 | 2 | 1 | 0.22 |
| Suratgarh | 0.846 | 0.182 | 0.105 | 0.18 | 0.11 | 3 | 2 | 0 | 0 | 0.31 | 2 | 1 | 0.23 |
| Chandrapur (DVC) | 0.825 | 0.212 | 0.122 | 0.21 | 0.12 | 3 | 2 | 0 | 0 | 0.37 | 2 | 1 | 0.28 |
| Metur | 0.816 | 0.226 | 0.064 | 0.23 | 0.06 | 3 | 2 | 0 | 0 | 0.32 | 2 | 1 | 0.28 |
| Khaparkheda | 0.798 | 0.252 | 0.106 | 0.25 | 0.11 | 3 | 2 | 0 | 0 | 0.39 | 2 | 1 | 0.32 |
| Sanjay Gandhi | 0.798 | 0.253 | 0.108 | 0.25 | 0.11 | 3 | 2 | 0 | 0 | 0.38 | 2 | 1 | 0.3 |
| Panipat | 0.796 | 0.256 | 0.095 | 0.26 | 0.1 | 3 | 2 | 0 | 0.26 | 0.1 | 3 | 2 | 0 |
| Korba west | 0.795 | 0.259 | 0.054 | 0.26 | 0.05 | 3 | 2 | 0 | 0 | 0.35 | 2 | 1 | 0.31 |
| Chandrapur (Maharashtra) | 0.793 | 0.261 | 0.153 | 0.26 | 0.15 | 3 | 2 | 0 | 0.26 | 0.15 | 3 | 2 | 0 |
| Raichur | 0.781 | 0.280 | 0.097 | 0.28 | 0.1 | 3 | 2 | 0 | 0.25 | 0.13 | 1 | 1 | 0.04 |
| Wanakbori | 0.757 | 0.322 | 0.083 | 0.32 | 0.08 | 3 | 2 | 0 | 0.32 | 0.08 | 3 | 2 | 0 |
| Uaki | 0.741 | 0.349 | 0.090 | 0.35 | 0.09 | 3 | 2 | 0 | 0.35 | 0.09 | 3 | 2 | 0 |
| Parli | 0.687 | 0.456 | 0.081 | 0.46 | 0.08 | 3 | 2 | 0 | 0.46 | 0.08 | 3 | 2 | 0 |
| Gandhi Nagar | 0.675 | 0.482 | 0.072 | 0.48 | 0.07 | 3 | 2 | 0 | 0.48 | 0.07 | 3 | 2 | 0 |
| GNDTPS (Bhatinda) | 0.652 | 0.533 | 0.102 | 0.53 | 0.1 | 3 | 2 | 0 | 0.33 | 0.32 | 1 | 1 | 0.24 |
| Koradi | 0.602 | 0.661 | 0.095 | 0.66 | 0.09 | 3 | 2 | 0 | 0.66 | 0.09 | 3 | 2 | 0 |
| Bhusawal | 0.589 | 0.697 | 0.000 | 0.7 | 0 | 3 | 3 | 0 | 0.7 | 0 | 3 | 3 | 0 |
| Satpura | 0.547 | 0.827 | 0.059 | 0.83 | 0.06 | 3 | 2 | 0 | 0.83 | 0.06 | 3 | 2 | 0 |
| Sikka | 0.517 | 0.935 | 0.094 | 0.94 | 0.09 | 3 | 2 | 0 | 0.94 | 0.09 | 3 | 2 | 0 |
| Rajghat | 0.493 | 1.027 | 0.103 | 1.03 | 0.1 | 3 | 2 | 0 | 1.03 | 0.1 | 3 | 2 | 0 |
| Ennore | 0.269 | 2.712 | 0.102 | 2.71 | 0.1 | 3 | 2 | 0 | 2.71 | 0.1 | 3 | 2 | 0 |

Note: The plants are listed in the descending order of their $b_y$ values. The shaded regions demarcate the three performance categories
*Col. stands for column of Table 1 and Row stands for row of Table 1.

| | Output-based FGL index | | | Graph Efficiency Index | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | $w_y$=1/3,$w_z$=2/3 | | | | | $w_y$=0, $w_z$=1 | | | | |
| **Plant name** | **$b_y$** | **$\beta_y$-1** | **1-$\beta_z$** | **$\theta$** | **$\gamma$** | **Row** | **Col*** | **$\delta_z$** | **$\theta$** | **$\gamma$** | **Row** | **Col*** | **$\delta_z$** |
| Bokaro B | 1 | 0 | 0.160 | 0 | 0.16 | 4 | 2 | 0 | 0 | 0.11 | 4 | 2 | 0 |
| Dahanu | 1 | 0 | 0.106 | 0 | 0.11 | 4 | 2 | 0 | 0 | 0.1 | 4 | 2 | 0 |
| Korba | 1 | 0 | 0.102 | 0 | 0.1 | 4 | 2 | 0 | 0 | 0.16 | 4 | 2 | 0 |
| Vindhyanchal | 1 | 0 | 0.100 | 0 | 0.1 | 4 | 2 | 0 | 0 | 0.1 | 4 | 2 | 0 |
| Singrauli | 0.999 | 0.001 | 0.106 | 0 | 0.11 | 2 | 1 | 0.01 | 0 | 0.11 | 2 | 1 | 0.01 |
| Ramagundem | 0.998 | 0.002 | 0.105 | 0 | 0.11 | 2 | 1 | 0 | 0 | 0.11 | 2 | 1 | 0 |
| Sipat | 0.991 | 0.009 | 0.105 | 0 | 0.11 | 2 | 1 | 0.01 | 0 | 0.11 | 2 | 1 | 0.01 |
| Simhadri | 0.983 | 0.017 | 0.096 | 0 | 0.12 | 2 | 1 | 0.03 | 0 | 0.12 | 2 | 1 | 0.03 |
| Rihand | 0.978 | 0.023 | 0.109 | 0 | 0.14 | 2 | 1 | 0.03 | 0 | 0.14 | 2 | 1 | 0.03 |
| Talcher | 0.972 | 0.029 | 0.085 | 0 | 0.12 | 2 | 1 | 0.04 | 0 | 0.12 | 2 | 1 | 0.04 |
| Budge Budge | 0.970 | 0.030 | 0.102 | 0 | 0.14 | 2 | 1 | 0.04 | 0 | 0.14 | 2 | 1 | 0.04 |
| Dadri (NCTPP) | 0.964 | 0.037 | 0.102 | 0 | 0.15 | 2 | 1 | 0.05 | 0 | 0.15 | 2 | 1 | 0.05 |
| Dr. N. Tata Rao | 0.955 | 0.047 | 0.103 | 0 | 0.16 | 2 | 1 | 0.07 | 0 | 0.16 | 2 | 1 | 0.07 |
| Unchahar | 0.951 | 0.052 | 0.108 | 0 | 0.17 | 2 | 1 | 0.07 | 0 | 0.17 | 2 | 1 | 0.07 |
| Kakatiya | 0.931 | 0.075 | 0.100 | 0 | 0.2 | 2 | 1 | 0.11 | 0 | 0.2 | 2 | 1 | 0.11 |
| Tuticorin | 0.927 | 0.079 | 0.095 | 0 | 0.21 | 2 | 1 | 0.13 | 0 | 0.21 | 2 | 1 | 0.13 |
| Farakka | 0.925 | 0.081 | 0.101 | 0 | 0.19 | 2 | 1 | 0.1 | 0 | 0.19 | 2 | 1 | 0.1 |
| Rayalseema | 0.924 | 0.083 | 0.107 | 0 | 0.21 | 2 | 1 | 0.12 | 0 | 0.21 | 2 | 1 | 0.12 |
| Kota | 0.923 | 0.084 | 0.105 | 0 | 0.23 | 2 | 1 | 0.14 | 0 | 0.23 | 2 | 1 | 0.14 |
| Kahalgaon | 0.914 | 0.094 | 0.099 | 0 | 0.19 | 2 | 1 | 0.1 | 0 | 0.19 | 2 | 1 | 0.1 |
| Kothagundem | 0.894 | 0.119 | 0.158 | 0 | 0.29 | 2 | 1 | 0.16 | 0 | 0.29 | 2 | 1 | 0.16 |
| IB Valley | 0.889 | 0.125 | 0.095 | 0 | 0.24 | 2 | 1 | 0.17 | 0 | 0.24 | 2 | 1 | 0.17 |
| Chhabra | 0.873 | 0.145 | 0.181 | 0 | 0.34 | 2 | 1 | 0.19 | 0 | 0.34 | 2 | 1 | 0.19 |
| North Chennai | 0.870 | 0.149 | 0.084 | 0 | 0.26 | 2 | 1 | 0.19 | 0 | 0.26 | 2 | 1 | 0.19 |
| Ropar | 0.857 | 0.166 | 0.094 | 0 | 0.29 | 2 | 1 | 0.21 | 0 | 0.29 | 2 | 1 | 0.21 |
| GHTPS (Lehra Mohabbat) | 0.850 | 0.177 | 0.105 | 0 | 0.3 | 2 | 1 | 0.22 | 0 | 0.3 | 2 | 1 | 0.22 |
| Nasik | 0.848 | 0.179 | 0.097 | 0 | 0.3 | 2 | 1 | 0.22 | 0 | 0.3 | 2 | 1 | 0.22 |
| Suratgarh | 0.846 | 0.182 | 0.105 | 0 | 0.31 | 2 | 1 | 0.23 | 0 | 0.31 | 2 | 1 | 0.23 |
| Chandrapur (DVC) | 0.825 | 0.212 | 0.122 | 0 | 0.37 | 2 | 1 | 0.28 | 0 | 0.37 | 2 | 1 | 0.28 |
| Metur | 0.816 | 0.226 | 0.064 | 0 | 0.32 | 2 | 1 | 0.28 | 0 | 0.32 | 2 | 1 | 0.28 |
| Khaparkheda | 0.798 | 0.252 | 0.106 | 0 | 0.39 | 2 | 1 | 0.32 | 0 | 0.39 | 2 | 1 | 0.32 |
| Sanjay Gandhi | 0.798 | 0.253 | 0.108 | 0 | 0.38 | 2 | 1 | 0.3 | 0 | 0.38 | 2 | 1 | 0.3 |
| Panipat | 0.796 | 0.256 | 0.095 | 0 | 0.32 | 2 | 1 | 0.25 | 0 | 0.32 | 2 | 1 | 0.25 |
| Korba west | 0.795 | 0.259 | 0.054 | 0 | 0.35 | 2 | 1 | 0.31 | 0 | 0.35 | 2 | 1 | 0.31 |
| Chandrapur (Maharashtra) | 0.793 | 0.261 | 0.153 | 0 | 0.36 | 2 | 1 | 0.24 | 0 | 0.36 | 2 | 1 | 0.24 |
| Raichur | 0.781 | 0.280 | 0.097 | 0 | 0.36 | 2 | 1 | 0.29 | 0 | 0.36 | 2 | 1 | 0.29 |
| Wanakbori | 0.757 | 0.322 | 0.083 | 0 | 0.36 | 2 | 1 | 0.3 | 0 | 0.36 | 2 | 1 | 0.3 |
| Uaki | 0.741 | 0.349 | 0.090 | 0 | 0.38 | 2 | 1 | 0.32 | 0 | 0.38 | 2 | 1 | 0.32 |
| Parli | 0.687 | 0.456 | 0.081 | 0 | 0.46 | 2 | 1 | 0.41 | 0 | 0.46 | 2 | 1 | 0.41 |
| Gandhi Nagar | 0.675 | 0.482 | 0.072 | 0 | 0.44 | 2 | 1 | 0.4 | 0 | 0.44 | 2 | 1 | 0.4 |
| GNDTPS (Bhatinda) | 0.652 | 0.533 | 0.102 | 0 | 0.49 | 2 | 1 | 0.43 | 0 | 0.49 | 2 | 1 | 0.43 |
| Koradi | 0.602 | 0.661 | 0.095 | 0.26 | 0.39 | 1 | 1 | 0.33 | 0 | 0.52 | 2 | 1 | 0.47 |
| Bhusawal | 0.589 | 0.697 | 0.000 | 0 | 0.5 | 2 | 1 | 0.5 | 0 | 0.5 | 2 | 1 | 0.5 |
| Satpura | 0.547 | 0.827 | 0.059 | 0.08 | 0.58 | 1 | 1 | 0.55 | 0 | 0.61 | 2 | 1 | 0.59 |
| Sikka | 0.517 | 0.935 | 0.094 | 0.94 | 0.09 | 3 | 2 | 0 | 0 | 0.53 | 2 | 1 | 0.48 |
| Rajghat | 0.493 | 1.027 | 0.103 | 0.83 | 0.26 | 1 | 1 | 0.18 | 0 | 0.6 | 2 | 1 | 0.55 |
| Ennore | 0.269 | 2.712 | 0.102 | 2.71 | 0.1 | 3 | 2 | 0 | 0 | 0.79 | 2 | 1 | 0.76 |

Note: The plants are listed in the descending order of their $b_y$ values. The shaded regions demarcate the three performance categories
*Col. stands for column of Table 1 and Row stands for row of Table 1.