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Abstract

We develop a tractable model of economic growth for the study of redistributive fiscal policy in a contemporary dual economy characterised by an abundance of unskilled labour in the (informal) rural agricultural sector capable of migrating and contributing to the human capital in the (formal) aggregated urban manufacturing and services sector by investing in costly skill acquisition, the essentiality of consumption in excess of subsistence level of an aggregated agricultural good, differences in the demand patterns of goods produced in the formal and informal sectors, and endogenous technological progress in both sectors. Macroeconomic tax equilibria induced by different configurations of fiscal policy instruments are derived. Economic growth will generally be unbalanced and fiscal policy has the potential to mitigate the resulting inequalities. The social welfare maximisation problem for selection of the optimal fiscal policy is posed. In a sequel to this work (Das and Murty (2022)), this model is employed to conduct some numerical simulations to study the features of socially optimal fiscal policies in such a dual economy.

Deriving optimal fiscal policies in a growing dual economy

1 Introduction.

In his well-known work, Lewis (1954) studied the dualism between a rural subsistence sector characterised by excess population with low (near zero) marginal productivities of labour and a growing urban capitalistic sector using modern technology in an underdeveloped economy. The wages in the capitalistic sector are pegged down till a point where the entire surplus labour in the subsistence sector is absorbed. The growth in the modern sector through reinvestments of profits so generated is conjectured to absorb the surplus (under-employed) labour in the subsistence sector. But Todaro (1969) argued that far from leading to equalisation of wages across the two sectors, the rural-urban migration leads to massive urban unemployment and the creation of an urban informal sector. Singer (1970) discusses how the lopsided nature of development in the modern manufacturing sectors of underdeveloped economies was responsible for large scale urban unemployment witnessed in such economies. Technologies and know-how in the modern sector were characterised by high capital intensities as they were either directly imported or borrowed from the West rather than indigenously developed based on the natural endowments and requirements of the labour-rich underdeveloped economies.

In this work, we abstract from the problem of large scale urban unemployment as studied by Todaro (1969) and Singer (1970). Rather, we focus just on the nature of dualism that exists between the rural agricultural sector and the urban manufacturing and service sectors in a contemporary developing economy. To the extent that such economies continue to be characterised by an abundance of unskilled-labour endowment, we believe that there are bound to be qualitative differences in the types of labour, capital, and technologies employed in these two sectors. Coupled with the differences in the pattern of demands for the goods produced in these sectors and the fact that the consumption of the informal/agricultural good above a subsistence level is essential for all households, these imply a persistence of the phenomenon of dualism in such economies. While the essentiality of the agricultural good implies that both the sectors have to co-exist, our intuition suggests that demand side factors will tend to depress the growth of the informal agricultural sector relative to the formal sector. This is in contrast to Lewis (1954) who did not incorporate demand-side factors in his model of a dual economy. Moreover, unlike in his model, such a dualism could co-exist with endogenous technological progress *and* government infrastructural support to improve production in both the sectors;[1] albeit the resulting growth in the two sectors will generally turn out to be unbalanced. Redistributive fiscal policies can play a big role in mitigating the inhibitive impacts of such a dualism on social welfare.

In this paper, we attempt to construct a tractable dynamic model of a modern dual economy with migration, whose equilibrium can be solved by adapting and extending the methodology developed in Turnovsky (1996). The social welfare maximisation problem for selection of the optimal fiscal policies is posed. In a sequel working paper (Das and Murty (2021)), this model is employed to conduct some numerical simulations to study the features of socially optimal fiscal policies in such a dual economy.

In his text, Acemoglu (2009) points out that dual economies are characterised by limited interactions between the modern and traditional sectors. In our model, these limited interactions manifest themselves in the form of restrictions on mobility of factors of production between the rural agricultural sector (referred to henceforth as the informal sector) and the aggregated urban manufacturing and services sector (referred to henceforth as the formal sector). In particular, technologies of contemporary labour-surplus developing economies require relatively skilled labour in the formal sector; while the rural agricultural sector is not so demanding with respect to the skill requirements of its labour force – labour employed in this sector is primarily unskilled. We view and model the skilled labour employed by the formal sector as standard human capital, accumulation of which can generate economic growth in this sector. Rural-urban migration in our model takes the form of imperfect mobility of labour between sectors. Migration has to be accompanied by costly investment in skill formation by the rural labour-force for them to be absorbed in the formal sector. The extent of migration and skill formation by the migratory labour force will be shown later to be obtained as choices of households that maximise their inter-temporal utilities.

Our model of a contemporary dual economy can be employed to study second-best fiscal policies that aim at redistribution to promote social equity as well as provision of infrastructural support to production in the two sectors and a standard public good that is consumed. As is empirically observed in many developing economies such as India, there is no taxation of incomes generated in the informal agricultural sector. All households (rural or urban), are however subjected to a consumption tax. Both forms of capital – physical and human– can be taxed in the formal sector. In addition, we assume that the economic rents generated by

[1] The green revolution in India during the 1960s along with governmental irrigation infrastructural support are examples of technological progress and government support to the rural agricultural sector.

governmental infrastructural expenditures in the formal sector can also be taxed.

Contemporary developing economies such as India are known to be experiencing the phenomenon of a "demographic dividend," whereby the economy witnesses a predominance of working-age population relative to its dependent population that is sustained for a long period of time. India has been experiencing a demographic dividend since 2005-06. According to the Economic Survey (2018-19), in India, the share of population in the predominantly working age group (20 to 59 years) was 50.5% in 2011 and has been increasing. It is projected to peak to 58.9% in 2041. With fewer people that are dependent on care and support and a large working population, there is a potential for expansion of the economic resources and their increased availability for bringing forth rapid economic growth. However, realising this potential requires a strong and wise institutional presence for facilitating greater productive investment and for designing and implementing other economic policies that are conducive for growth.

Our work can be placed in the context of the phenomenon of a demographic dividend, because we feel that the structure of the economy that can tap this dividend successfully will have features of the the dual economy that we study. In particular, as in this work, such an economy will be labour rich with a sustained increase in its working population, which will be predominantly unskilled to begin with. Designing appropriate fiscal policies is crucial for tapping into the demographic-dividend induced increases in consumption and income-tax base to raise greater tax revenue for financing greater expenditure on productive public infrastructure and greater redistribution for mitigating income inequalities brought about by unbalanced sectoral growths. Trade-offs arise between the government's objectives of redistribution and economic growth, which will also have to balanced by sound fiscal policies. Further, fiscal policies can also play a big role in influencing the extent of migration and skill acquisition by unskilled labour force, thus leading to more efficient allocation of skilled and unskilled labour force across sectors in such economies.

There is significant literature that highlights and studies the commonness in the process of structural change witnessed by many currently developed countries, where the relative importance of the agricultural sector declined and that of the manufacturing and services sectors rose over time. Many works in this literature have been comprehensively reviewed by Gabardo et al (2017). However, it seems that the study and analysis of a dual economy is mostly limited to that provided by Lewis (1954). The dualism phase seems to be seen in much of this literature as a mere predecessor of rapid economic growth, which the currently developed countries soon and very successfully left behind as they transitioned from periods of high population growths and low savings rates to periods of low birth and high investment rates. More importantly, this literature generally tends to abstract from the study of the role of fiscal policy in smoothening out inequalities and promoting economic growth during this transition.

On the other hand, the scant literature that does study fiscal policies in economies with formal and informal sectors does not address the phenomenon of dualism. In Penelosa and Turnovsky (2005) the formal sector is assumed to employ a more capital intensive technology and only incomes generated in this sector are subject to taxation. All households receive a uniform transfer; thus, the transfer is not redistributive. There is only one consumption good that can be produced either in the formal or in the informal sector, so differences in demand side factors for goods produced in the formal and informal sectors are not studied. Further, in contrast to our work that assumes imperfect mobility of labour and capital across sectors, which makes our economy a dual one with unbalanced growth, there is perfect mobility of labour and capital between the two sectors in Penelosa and Turnovsky (2005) so that a macroeconomic equilibrium in their model is characterised by a balanced growth. Their work is concerned with studying the efficacy of capital taxation and shows that, given sectoral differences in the capital intensities of technologies, second-best optimal fiscal policies may require significant taxation of capital in addition to labour to raise a fixed amount of revenue for the government.

In Section 2, we distinguish between formal (urban manufacturing and services) and informal (rural agricultural) sector households and describe their preferences; the laws of motions of human and physical capital, as well as of the net unskilled labour force supplied by these households; and its instantaneous budget constraint. In Section 3, we pose the inter-temporal welfare maximisation problems of the two types of households and derive all the first order conditions of this problem and provide their interpretations. Section 4 discusses the production side of this economy, while Section 5 describes the government's budget balance and its implications for market clearing that follows from the Walras law. Section 6 solves for the decentralised macro-economic equilibrium of the model for every configuration of tax rates. In particular, the long-run growth rates of various economic variables are derived, and it is shown that unbalanced sectoral growth will generally be true in such economies. Section 7 derives an iso-elastic inter-temporal social welfare function. The social welfare maximisation problem is posed along with all the relevant constraints for deriving the optimal fiscal policy. We conclude in Section 8. In a sequel to this work (Das and Murty (2022)), the theory developed here is employed to perform some numerical simulations to get a flavour of the characteristics of the social-welfare maximising fiscal policies that can be generated by our model.

2 Households.

In modern times, with removal of social and physical restrictions on occupational mobility and greater access to education, many households possess members that work both in the formal and informal sectors. Some members of a primarily rural household may have acquired skills that make it possible for them to work in the formal sector, but they continue to pool their incomes with other unskilled members of the rural household who work in the informal, in our case, the agricultural sector. Similarly, an urban household with primarily skilled workers may also have acquired working assets which are productive in the informal sector. Our general model allows these features, but distinguishes between a *primarily* rural/informal-sector household from a *primarily* urban/formal-sector household.

There are two representative households in this economy indexed by $m = 1, 2$. Each household has resources that can be employed in the formal or the informal sector. The formal-sector specific human and physical capital possessed by household m are denoted by K_{1m} and H_m, respectively; while its possession of physical capital and the (gross) unskilled labour resource that are employed in the informal sector are denoted by K_{2m} and l_m, respectively.[2] In addition, both households can also accumulate wealth in the form of another asset that is exogenously supplied and yields unvarying returns. This is denoted by B_m for $m = 1, 2$. Since this turns out to be analytically analogous to a foreign bond in Turnovsky (1996), we will call this asset a private bond or simply a bond.[3]

We will call the first household, *i.e.,* $m = 1$, a representative household in the formal sector, while the second household, *i.e.,* $m = 2$, will represent a household in the informal sector. The two households are distinguished by the fact that the formal sector household initially possesses disproportionately larger shares of the physical and human capital resources used in the formal sector, while the informal sector household initially holds disproportionately larger shares of the physical and unskilled labour resources used in the informal sector.

The initial endowments of bonds, capital–both physical and human, and unskilled labour of household m are $B_m(0)$, $K_{1m}(0)$, $K_{2m}(0)$, $H_m(0)$, and $l_m(0)$. Under our assumptions, $K_{11}(0) > K_{12}(0)$, $K_{22}(0) > K_{21}(0)$, $H_1(0) > H_2(0)$, and $l_2(0) > l_1(0)$. The model includes the special cases of *pure formal* or *pure informal* representative households. In the former case, the formal

[2]It is assumed that there are differences in the nature of physical capital used for production in the formal and informal sectors – physical capital is not substitutable between the two sectors. Moreover, in the next section, Section 2.1, we also differentiate between gross and net unskilled labour force employed in the informal sector.

[3]Intuitively, perhaps this asset can be like gold, with the return on this asset (which is assumed to be unvarying in this analysis) being analogous to the price of gold.

sector household possesses no resources used in the informal sector, *i.e.,* $K_{21}(0) = l_1(0) = 0$; while in the latter case, the informal sector household initially possesses no resources used in the formal sector, *i.e.,* $K_{12}(0) = H_2(0) = 0$.

While the unskilled labour resource grows exogenously at an exponential rate n, both types of physical capital and the human capital resources grow endogenously in our model.

2.1 Migration in a dual economy and laws of motion of physical capital, human capital, and net unskilled labour force in the informal sector.

The goods produced by the formal and informal sectors are distinct with respect to consumer preferences for these goods, which will be described in Section 2.2.

Rural-urban migration in our model takes the form of imperfect mobility of labour between sectors. The extent of migration of unskilled labour force in household m is denoted by $X_m \geq 0$, while $\beta_m \geq 0$ denotes the average amount of education/skill acquired by the unskilled labour force that decides to migrate in household m.

Given the migration of labour force from the informal to the formal sector, the law of motion of *net* unskilled labour force in household m (denoted by L_m) that is available to work in the informal sector in household m becomes relevant. To derive this for the continuous time case, lets first consider the more intuitive discreet time case, where the law of motion of L_m is given by

$$\begin{aligned} L_m(t+1) &= L_m(t) + l_m(t+1) - l_m(t) - X_m(t) \\ \implies L_m(t+1) - L_m(t) &= l_m(t+1) - l_m(t) - X_m(t) \\ &= l_m(t)(1+n) - l_m(t) - X_m(t) \\ &= nl_m(t) - X_m(t), \end{aligned}$$

where the first equality indicates that the net unskilled labor force contributed by household m at time period $t+1$ is its net unskilled labor force at time period t plus addition to gross unskilled labour force between time period t and $t+1$ given by $l_m(t+1) - l_m(t)$ net of migration that happened between period t and $t+1$, denoted by $X_m(t)$.

The last equality shows that the addition to the net unskilled labour force between time period t and $t+1$ is equal to the addition to gross unskilled labour force between time period

t and $t+1$ minus the migration that happened between period t and $t+1$.

Note, in particular

$$L_m(1) = L_m(0) + nl_m(0) - X_m(0)$$

We assume that the initial stock of net unskilled labour in household m, $L_m(0)$, is given and that $L_m(0) \leq l_m(0)$. The continuous time analogue of the law of motion of L_m can be written as

$$\dot{L}_m(t) = nl_m(t) - X_m(t), \quad \text{where } L_m(0) \text{ is given.}$$

The laws of motion for state variables H_m, K_{1m} and K_{2m} that denote the human and physical capital accumulation by household m are given, respectively, by:

$$\begin{aligned}
\dot{H}_m(t) &= I_{hm}(t) + \beta_m X_m(t) \\
\dot{K}_{1m}(t) &= I_{k_1 m}(t) - \delta_{k_1} K_{1m}(t) \\
\dot{K}_{2m}(t) &= I_{k_{2m}}(t) - \delta_{k_2} K_{2m}(t),
\end{aligned}$$

where I_{hm}, $I_{k_1 m}$, and $I_{k_2 m}$ denote investments in human capital and the two types of physical capital, respectively. The depreciation rates of physical capital used in the formal and informal sectors are denoted by δ_{k_1} and δ_{k_2}, respectively.

The contribution of migratory unskilled labour force of household m to its human capital stock through acquisition of skill is denoted by $\beta_m X_m$, where β is an abstract measure that can be intuitively interpreted as the skill/education factor. The higher is β_m the more is added to the human capital stock by one unit of migrating labour. For example, if $\beta_m = 0$, then the migrating labour will add nothing to the stock of human capital of household m. If $\beta_m = 1$, then one unit of migrating labour is able to contribute an additional unit to the stock of human capital. If $\beta_m > 1$ (resp., less than one), then one unit of migrating labour is able to contribute more than one unit (resp., less than one) to the stock of human capital.

To the best of our knowledge, the dynamics of the unskilled labour supply available for production in the agricultural sector and migration presented in our model are novel. The net-of-migration labour supply to the agricultural sector evolves organically in the model and has properties of a state variable in our dynamic framework. Unlike in the text by Acemoglu (2009),[4] where the barriers on mobility of labour/migration are modelled in a reduced form way by a parameter that captures the speed of migration, in our model the restrictions to mobility

[4]See Chapter 21 of the text.

of labour arises on account of the cost the migrating labour force has to incur to acquire skill to contribute to human capital in the formal sector.

In our model, the physical capital employed by the formal sector is also assumed to be qualitatively different from that employed in the informal sector. We feel that this is true of many developing economies, where technologies employed in the rural agricultural and urban manufacturing and services sector are significantly different, so that there is no costless sharing/ allocation of aggregate capital between the two sectors as modelled in most of the literature on structural transformation, which mainly studies the historical experiences of currently developed economies.[5] Nevertheless, as will be seen in Section 2.3, both types of physical capital will compete for investment expenditure out of incomes of households.

2.2 Household preferences.

Let C_{1m} and C_{2m} denote household m's consumption of the formal and informal sector goods, respectively. Let G_c denote non-rival consumption of a public good. The instantaneous utility function of the households is

$$U\left(C_{1m}, C_{2m}, G_c\right) = \frac{1}{1-\epsilon}\, c\left(C_{1m}, C_{2m}\right)^{1-\epsilon} G_c^{1-\epsilon}, \qquad \epsilon > 0,\ \epsilon \neq 1,$$

The Engels' law states that the proportion of income that a household spends on food (agricultural products) decreases and that spent on other goods (manufactured goods and services) increases as its income increases. Thus, the patterns of demands for goods produced in the formal and informal sectors are distinctly different – the income elasticity of demand for agricultural products will be low – less than one, while it will be higher for the other goods. At the same time we assume that the consumption of the agricultural goods over and above a minimum subsistence level is essential for living – there is no life if this is zero. This feature of preferences can be captured by adopting a non-homothetic Stone-Geary form of preferences.[6] In this case, the function $c\left(C_{1m}, C_{2m}\right)$ has the following form:

$$c_m = c\left(C_{1m}, C_{2m}\right) = \left(C_{1m} + \mu_1\right)^{\eta_1} \left(C_{2m} - \mu_2\right)^{\eta_2}$$

with $\mu_1 > 0,\ \mu_2 > 0,\ C_{2m} \geq \mu_2,\ C_{1m} \geq 0,\ \ \eta_1 + \eta_2 = 1$. Here, μ_2 can be interpreted as the

[5] See, the survey article by Gabardo et al (2017) for some examples of such works.

[6] We borrow this form of preferences from works such as Kongsamut et al. (2001) that study structural change during the course of development of an economy that is induced by demand-side factors.

subsistence level of consumption of the agricultural/informal-sector good. Hence, the difference $C_{2m} - \mu_2$ denotes the consumption of the agricultural good over and above the subsistence level by the m^{th} household. It is clear that, given our Stone-Geary preference structure, consumption of the agricultural good over and above the subsistence level is an essential good. If this is zero, then welfare of household m is zero. On the other hand, μ_1 can be interpreted as the level of consumption of the formal sector good that is guaranteed to them even if they do not pay for it, so that $C_{1m} + \mu_1$ can be interpreted as the total consumption of the formal sector good. It is the sum of acquired consumption of the formal sector good, $C_{1m} \geq 0$, and the guaranteed consumption of this good. It is positive as long as $\mu_1 > 0$. The particular Stone-Geary preference structure adopted implies that demand for the formal sector good is elastic, while the demand for the informal sector good is inelastic. Intuitively, the informal sector can be thought of as including the non-cash crops producing part of the agricultural sector, demand for whose output is relatively inelastic, while the formal sector produces manufactured goods and services or cash crops in the agricultural sector whose demand is relatively more elastic.

2.3 The instantaneous budget constraint of a household.

Let w_1 and w_2 denote, respectively, the formal and informal sector wage rates; while r_1 and r_2 denote, respectively, the formal and informal sector returns on capital. The rate of return on bonds is denoted by r_b. We assume that it is a time-invariant exogenously-determined constant. Wage and capital incomes are subject to taxation only in the formal sector. In the real world the informal sector usually escapes income taxation. This could be either because of prevalence of large scale tax evasion in this sector due to difficulties faced by the government in verifying incomes of people working in this sector or because this sector may be constitutionally exempted from income taxation due to its under-developed nature. Let τ_w, τ_k, τ_b denote the tax rates on wage income, rental income, and income earned from bonds, respectively. In addition, at every time period t, we assume that the informal sector representative household receives a transfer $T(t) \geq 0$ from the government that adds to its income. Consumption of the formal sector good, whether by the formal or informal household, is subject to a tax at the rate τ_c, while there is no tax on the consumption of the informal sector good.[7] The price of the formal sector good is normalised to be one and the price of the informal sector good is denoted by p.

[7] A basic assumption in the public economics literature is that, due to information constraints, the government can only tax transactions and incomes that can be directly observed. It is assumed that transactions between consumers and the formal sector producers can be directly observed by the government, while the same cannot be said of the informal sector.

Investments in physical and human capital in the formal sector involve convex adjustment costs given, respectively, by $\frac{\pi I_{k_1 m}^2}{2K_{1m}}$ and $\frac{\theta I_{hm}^2}{2H_m}$, where $\pi > 0$ and $\theta > 0$ are fixed parameters. Cost of skill formation for labour force migrating from the informal to formal sector depends both on the extent of migration X_m and the average skill-level sought, which is denoted by $\beta_m \geq 0$. Skill formation costs are higher if β_m chosen by the household is higher or if X_m is higher. Skill formation costs are assumed to take a convex form given by $X_m\left(1 + \frac{S(\beta_m)X_m}{2L_m}\right)$ whenever $X_m > 0$. Function S is assumed to be increasing and strictly convex.[8] No adjustment cost is assumed for investment in physical capital in the informal sector.[9]

We will assume that workers are self-employed in the informal sector, *i.e.,* each household is also directly a producer with respect to its participation in the informal sector. The profit generated in the informal sector by a producer-household, denoted by Π_{2m}, goes back to the household in its role as a consumer.

Given the tax rates τ_w, τ_k, τ_b, and τ_c and a trajectory of transfers $T(t)$, the instantaneous budget constraint faced by household m can now be written as follows:

$$\begin{aligned}
\dot{B}_m(t) &= (1-\tau_w)w_1(t)H_m(t) + w_2(t)L_m(t) + (1-\tau_k)r_1(t)K_{1m}(t) + r_2(t)K_{2m}(t) \\
&\quad +(1-\tau_b)r_b B_m(t) - (1+\tau_c)C_{1m}(t) - p(t)C_{2m}(t) \\
&\quad -X_m(t)\left(1 + \frac{S(\beta_m(t))X_m(t)}{2L_m}\right) \\
&\quad -I_{k_1 m}(t)\left(1 + \frac{\pi I_{k_1 m}(t)}{2K_{1m}(t)}\right) - I_{k_{2m}(t)} - I_{hm}(t)\left(1 + \frac{\theta I_{hm}(t)}{2H_m(t)}\right) + T(t)\ Ind_m \\
&\quad +\Pi_{2m}(t)\ \Lambda\left(l_m(0), K_{2m}(0)\right) \\
Ind_m &= 0 \text{ for } m = 1 \\
&= 1 \text{ for } m = 2 \\
\Lambda\left(l_m(0), K_{2m}(0)\right) &= 0 \text{ when } l_m(0) = K_{2m}(0) = 0 \\
&= 1 \text{ otherwise} \qquad (1)
\end{aligned}$$

$\dot{B}_m(t)$ denotes the bond accumulation in period t by household m. The instantaneous budget constraint clearly reveals that this is the excess of total income of household m in period t over its consumption and investment expenditures in period t. The constraint is written in a general

[8]Additional weaker properties of function S that are sufficient for our analysis to hold will be provided later in Section 6.2.2.

[9]Perhaps because they could be assumed to be rudimentary and easier to set-up and operate as compared to the capital employed in the formal sector. A particular approach is adopted in Section 6.6 to solve the ensuing non-degeneracy in dynamics due to this assumption. Footnote 6 of Turnovsky (1999) explains the importance of the assumption of convex adjustment costs for investment in dynamic models of small open economies for generating non-degenerate dynamics.

form that encompasses cases of both the formal sector household ($m = 1$) and the informal sector household ($m = 2$). The formal sector household receives no transfer income, while the informal sector household does. The indicator function Ind_m captures this. Further, in the case of the formal sector household, the formulation can also distinguish between a pure formal sector household and a non-pure one. The pure formal sector household does not participate in informal activities, *i.e.*, $l_1(0) = K_{21}(0) = 0$, and hence receives no income including profit income from the informal sector. The indicator function Λ_m captures this.

3 Intertemporal welfare maximisation by households.

Household m solves the following intertemporal welfare maximisation problem:

$$
\begin{aligned}
\max \quad & \int_0^\infty U(C_{1m}(t), C_{2m}(t), G_c(t)) \exp\{-\rho t\} dt \\
& \text{subject to} \\
\dot{L}_m(t) &= nl_m(t) - X_m(t) \\
\dot{H}_m(t) &= I_{hm}(t) + \beta X_m(t) \\
\dot{K}_{1m}(t) &= I_{k_1 m}(t) - \delta_{k_1} K_{1m}(t) \\
\dot{K}_{2m}(t) &= I_{k_{2m}}(t) - \delta_{k_2} K_{2m}(t) \\
\dot{B}_m(t) &= (1-\tau_w) w_1(t) H_m(t) + w_2(t) L_m(t) + (1-\tau_k) r_1(t) K_{1m}(t) + r_2(t) K_{2m}(t) \\
& +(1-\tau_b) r_b B_m(t) - (1+\tau_c) C_{1m}(t) - p(t) C_{2m}(t) \\
& -X_m(t)\left(1 + \frac{S(\beta_m(t)) X_m(t)}{2L_m}\right) \\
& -I_{k_1 m}(t)\left(1 + \frac{\pi I_{k_1 m}(t)}{2K_{1m}(t)}\right) - I_{k_{2m}(t)} - I_{hm}(t)\left(1 + \frac{\theta I_{hm}(t)}{2H_m(t)}\right) + T(t)\ Ind_m \\
& +\Pi_{2m}(t)\ \Lambda_m\left(l_m(0), K_{2m}(0)\right) \\
& \lim_{t\to\infty} B(t) e^{(1-\tau_b) r_b t} \geq 0 \\
\forall\ t \quad & K_{1m}(t) \geq 0,\ K_{2m}(t) \geq 0,\ H_m(t) \geq 0,\ L_m(t) \geq 0,\ X_m(t) \geq 0, \\
& C_{1m}(t) \geq 0,\ C_{2m}(t) \geq 0,\ I_{k_1 m}(t) \geq 0,\ I_{k_{2m}}(t) \geq 0,\ \beta(t) \geq 0. \\
& B_m(0),\ K_{1m}(0),\ K_{2m}(0),\ H_m(0),\ l_m(0),\ \text{and } L_m(0) \text{ are all given}
\end{aligned}
$$

Denoting the co-state variables corresponding to the laws of motion of B_m, K_{1m}, K_{2m}, and H_m in the above problem by λ_{bm} , $\lambda_{k_1 m}$, $\lambda_{k_2 m}$, and λ_{hm}, respectively, the present value

Hamiltonian for the above problem is

$$\mathcal{H} = U(C_{1m}, C_{2m}, G_c).e^{-\rho t} + \lambda_{bm} e^{-\rho t} \Bigg[(1-\tau_w) w_1 H_m + w_2 L_m + (1-\tau_k) r_1 K_{1m} + r_2 K_{2m}$$
$$+ (1-\tau_b) r_b B_m - (1+\tau_c) C_{1m} - p C_{2m} - X_m \left(1 + S(\beta_m)\frac{X_m}{2L_m}\right) - I_{k_1 m}\left(1 + \pi \frac{I_{mk1}}{2K_{1m}}\right)$$
$$- I_{K_{2m}} - I_{hm}\left(1 + \theta . \frac{I_{hm}}{2H_m}\right) + T\ Ind_m$$
$$+ \Pi_{2m}\ \Lambda\left(l_m(0), K_{2m}(0)\right) - \dot{B}_m \Bigg]$$
$$+ \lambda_{k_1 m} e^{-\rho t}(I_{k_1 m} - \delta_{k_1} K_{1m} - \dot{K}_{1m}) + \lambda_{k_2 m} e^{-\rho t}(I_{k_{2m}} - \delta_{k_2} K_{2m} - \dot{K}_{2m})$$
$$+ \lambda_{hm} e^{-\rho t}(I_{hm} + \beta_m X_m - \dot{H}_m) + \lambda_{lm} e^{-\rho t}(n l_m - X_m - \dot{L})$$

The following first order conditions are derived assuming an interior solution to the above problem.

3.1 First-order conditions with respect to the control variables.

The first-order conditions with respect to C_{1m} and C_{2m} are

$$\begin{aligned} \frac{\partial \mathcal{H}}{\delta C_{1m}} = 0 \implies \frac{\partial U}{\partial C_{1m}} &= \lambda_{bm}(1+\tau_c) \\ \frac{\partial \mathcal{H}}{\delta C_{2m}} = 0 \implies \frac{\partial U}{\partial C_{2m}} &= \lambda_{bm} p, \end{aligned} \tag{2}$$

which yield the standard equalization of the marginal rates of substitution between consumptions of goods 1 and 2 at any time point to the ratio of the consumer (inclusive of tax) prices for these goods.[10]

$$\frac{\frac{\partial U}{\partial C_{1m}}}{\frac{\partial U}{\partial C_{2m}}} = \frac{(1+\tau_c)}{p} \tag{3}$$

We denote the shadow prices of human capital, net unskilled labour, and physical capital in the formal and informal sectors relative to the shadow price of private bond by λ_{hbm}, λ_{lbm}, $\lambda_{k_1 bm}$, and $\lambda_{k_2 bm}$, respectively.[11] the following first-order conditions with respect to migration, skill formation, investments in capital (the two types of physical capital and human capital) reflect

[10]Recall that good 1 is the numeraire and good 2 is not subject to a consumption tax and its price is denoted by p.

[11]These are analogous to the Tobin q. Thus, $\lambda_{hbm} = \frac{\lambda_{hm}}{\lambda_{bm}}$, $\lambda_{k_1 bm} = \frac{\lambda_{k_1 m}}{\lambda_{bm}}$, $\lambda_{k_2 bm} = \frac{\lambda_{k_2 m}}{\lambda_{bm}}$, $\lambda_{lbm} = \frac{\lambda_{lm}}{\lambda_{bm}}$.

the equalisation of the marginal cost to the marginal benefit from each of these variables:

$$\frac{\partial \mathcal{H}}{\delta X_m} = 0 \implies \lambda_{bm} e^{-\rho t}\left[-1 - \frac{S(\beta_m)}{2}.\frac{2X_m}{L_m}\right] + \lambda_{hm} e^{-\rho t}\beta_m - \lambda_{lm} e^{-\rho t} = 0 \tag{4}$$

At the household's inter-temporal optimum, at each time point, the marginal cost of migration, which includes the marginal cost of acquiring β_m level of skill by the migrating labour force to its transit to the formal sector and the shadow value of the loss of a unit of unskilled labour available to the informal sector due to migration, should be equal to the marginal benefit from migration due to acquisition of β_m amount of skill by the migrating unskilled labour joining the skilled labour force in the formal sector.

$$\frac{\partial \mathcal{H}}{\delta \beta_m} = 0 \implies \lambda_{bm} e^{-\rho t}\left[-\frac{{X_m}^2}{2L_m} S'(\beta_m)\right] + \lambda_{hm} X_m e^{-\rho t} = 0 \tag{5}$$

At the household's inter-temporal optimum, at each time point, the marginal cost of acquisition of skill by X_m amount of migrating unskilled labour force should be equal to the shadow value of the additional contribution to human capital in the formal sector when X_m amount of migratory labour acquire an additional unit of skill.

The first-order conditions (6), (7), and (8) imply that, at the household's inter-temporal optimum, at each time point, the marginal costs of investment in physical capital (whether employed in the formal or the informal sector) and the human capital are equal to their respective marginal benefits, which are the respective shadow values of an extra unit of physical or human capital employed. The marginal costs of investment in the physical and human capital in the case of the formal sector include both the purchase and the adjustment costs, while in the case of the physical capital employed in the informal sector, this includes only the purchase cost. Moreover, (6) and (8) show that investments as a proportion of the stocks of physical and human capital employed in the formal sector (which we shall also call as the rates of investment in physical and human capital employed in the formal sector) are increasing respectively in the relative shadow prices of physical and human capital employed in the formal sector. Further, (7) shows that the relative shadow price of the physical capital employed in the informal sector is a constant equal to one, *i.e.,* the shadow prices of the bond and physical capital employed in the informal sector are both equal.

$$\frac{\partial \mathcal{H}}{\delta I_{k_1 m}} = 0 \implies \lambda_{bm} e^{-\rho t}\left(-1 - \frac{\pi.2.I_{k_1 m}}{2.K_{1m}} \right) + \lambda_{k_1 m} e^{-\rho t} = 0$$
$$\implies \frac{I_{k_1 m}}{K_{1m}} = \frac{\lambda_{k_1 bm} - 1}{\pi} \tag{6}$$

$$\frac{\partial \mathcal{H}}{\delta I_{k_2 m}} = 0 \implies -\lambda_{bm} e^{-\rho t} + \lambda_{k_2 m} e^{-\rho t} = 0$$
$$\implies \frac{\lambda_{k_2 m}}{\lambda_{bm}} = \lambda_{k_2 bm} = 1 \tag{7}$$

$$\frac{\partial \mathcal{H}}{\delta I_{hm}} = 0 \implies \lambda_{bm} e^{-\rho t}\left(-1 - \frac{\theta.2.I_{hm}}{2.H_m} \right) + \lambda_{hm} e^{-\rho t} = 0$$
$$\implies \frac{I_{hm}}{H_m} = \frac{\lambda_{hbm} - 1}{\theta} \tag{8}$$

3.2 The first-order conditions with respect to the state variables.

The necessary condition for a consumer optimum with respect to the state variables have interpretations similar to those found in Turnovsky (1996). In particular, the condition with respect to the bond equates the after-tax returns from holding a bond to the the present discounted value of the capital-loss incurred when an additional bond is secured.

$$\frac{\partial \mathcal{H}}{\delta B_m} = -\frac{d}{dt}(\lambda_{bm} e^{-\rho t}) \quad \implies (1 - \tau_b) r_b = -\left(\frac{\dot{\lambda}_{bm}}{\lambda_{bm}} - \rho \right) \tag{9}$$

From this we infer that

$$\lambda_{bm}(t) = \lambda_{bm}(0) e^{(\rho - (1-\tau_b) r_b) t} \tag{10}$$

The first-order necessary conditions with respect to the state variables K_{1m}, K_{2m}, H_m, and L_m reflect the equation of the after-tax return from each of these variables to the after-tax return from the bond.

The presence of convex adjustment costs of investments in physical and human capital in the formal sector and the convex cost of skill acquisition for the migratory labour force imply that the after-tax return from each of these state variables includes (i) the capital gains (reflected in the increases in their shadow prices relative to the shadow price of the bond), (ii) reductions in the adjustment/skill-formation costs due to a unit increase in the level of the state variable, and (iii) the after-tax increase in output (measured in terms of the relative shadow value) due to a unit increase in usage of the state variable. Thus, for example, the first-order condition with respect to K_{1m} works out to be:

$$\frac{\partial \mathcal{H}}{\delta K_{1m}} = -\frac{d}{dt}(\lambda_{k_1 m} e^{-\rho t})$$

$$\implies \frac{\lambda_{bm}}{\lambda_{k_1 m}} \left[(1-\tau_k) r_1 + \frac{\pi {I_{k_1 m}}^2}{2.{K_{1m}}^2} \right] - \delta_{k_1} = \rho - \frac{\dot{\lambda}_{k_1 m}}{\lambda_{k_1 m}}$$

Substituting from (6), we have

$$\frac{\lambda_{bm}}{\lambda_{k_1 m}} \left[(1-\tau_k) r_1 + \frac{\pi}{2}.\left(\frac{\lambda_{k_1 bm} - 1}{\pi} \right)^2 \right] - \delta_{k1} = \rho - \frac{\dot{\lambda}_{k_1 m}}{\lambda_{k_1 m}}$$

$$\implies \frac{(1-\tau_k) r_1}{\frac{\lambda_{k_1 m}}{\lambda_{bm}}} + \frac{(\lambda_{k_1 bm} - 1)^2}{2\pi.\frac{\lambda_{k_1 m}}{\lambda_{bm}}} - \delta_{k1} = \rho - \frac{\dot{\lambda}_{k_1 m}}{\lambda_{k_1 m}},$$

which implies[12]

$$\frac{(1-\tau_k) r_1}{\lambda_{k_1 bm}} + \frac{\dot{\lambda}_{k_1 bm}}{\lambda_{k_1 bm}} + \frac{(\lambda_{k_1 bm} - 1)^2}{2\pi \lambda_{k_1 bm}} - \delta_{k1} = (1-\tau_b) r_b \tag{11}$$

Similarly too, we can show that

$$\frac{\partial \mathcal{H}}{\delta H_m} = -\frac{d}{dt}(\lambda_{hm} e^{-\rho t})$$

$$\implies \frac{(1-\tau_w) w_1}{\lambda_{hbm}} + \frac{\dot{\lambda}_{hbm}}{\lambda_{hbm}} + \frac{(\lambda_{hbm} - 1)^2}{2\theta \lambda_{hbm}} = (1-\tau_b) r_b \tag{12}$$

and

$$\frac{\partial \mathcal{H}}{\delta L_m} = -\frac{d}{dt}(\lambda_{l_m} e^{-\rho t})$$

$$\implies \frac{w_2 + S(\beta_m)\frac{{X_m}^2}{2{L_m}^2}}{\lambda_{lbm}} + \frac{\dot{\lambda}_{lbm}}{\lambda_{lbm}} = (1-\tau_b) r_b \tag{13}$$

In the absence of adjustment costs of investment in the physical capital employed in the

[12] The definition of $\lambda_{k_1 mb}$ implies that $\frac{\dot{\lambda}_{k_1 bm}}{\lambda_{k_1 bm}} = \frac{\dot{\lambda}_{k_1 m}}{\lambda_{k_1 m}} - \frac{\dot{\lambda}_{bm}}{\lambda_{bm}}$. Employing (9), this in turn implies that

$$\frac{\dot{\lambda}_{k_1 m}}{\lambda_{k_1 m}} = \frac{\dot{\lambda}_{k_1 bm}}{\lambda_{k_1 bm}} + \rho - (1-\tau_b) r_b$$

. This has been employed to derive (11).

informal sector, the first order condition with respect to K_{2m} precludes a capital gains component and directly equates the net increase in output due to usage of additional unit of this capital to the after-tax return from bond. Thus, we have

$$\begin{aligned}
\frac{\partial \mathcal{H}}{\delta K_{2m}} &= -\frac{d}{dt}(\lambda_{k_{2m}} e^{-\rho t}) \quad \Longrightarrow \quad \frac{r_2}{\lambda_{k_2 bm}} - \delta_2 = -\frac{\dot{\lambda}_{k_2 m}}{\lambda_{k_2 m}} + \rho \\
\Longrightarrow \frac{r_2}{\lambda_{k_2 bm}} - \delta_2 &= -\frac{\dot{\lambda}_{k_2 bm}}{\lambda_{k_2 bm}} - \frac{\dot{\lambda}_{bm}}{\lambda_{bm}} + \rho \\
\Longrightarrow r_2 - \delta_2 &= (1-\tau_b) r_b, \qquad (14)
\end{aligned}$$

where we have employed (7) and (9).

3.3 Laws of motion of state variables K_{1m}, H_m, and L_m.

It follows from the law of motion of K_{1m} that

$$\frac{\dot{K}_{1m}}{K_{1m}} = \frac{I_{k_1 m}}{K_{1m}} - \delta_{k_1},$$

which using (6) implies that

$$\frac{\dot{K}_{1m}}{K_{1m}} = \frac{\lambda_{k_1 bm} - 1}{\pi} - \delta_{k_1} =: \psi_{k_1 m} \qquad (15)$$

Similarly, using (8) and the law of motion of H_m, we obtain

$$\dot{H}_m = \left(\frac{\lambda_{hbm} - 1}{\theta} \right) H_m + \beta_m X_m \qquad (16)$$

The law of motion of L_m implies that

$$\frac{\dot{L}_m}{l_m} = n - \frac{X_m}{l_m}. \qquad (17)$$

4 Production

Technologies of both the formal and informal sectors will be assumed to be affine. An extension of the AK model of growth is employed, whereby the marginal products of capital and unskilled labour in the informal sector and the marginal products of both human and physical capital in the formal sector are assumed to be constant.[13] This extension facilitates greatly the use of techniques in Turnovsky (1996), which make the analysis of this otherwise complex growth model tractable.

In addition, we also model infrastructural aid from the government in both the sectors as two public goods that positively and linearly affect output production in both the formal and informal sectors. There are $\mathcal{I}$ firms in the formal sector. Since households are self employed in the informal sector, the number of production units in the informal sector is equal to the number of households engaging in production in the informal sector.[14]

For all firms $i = 1, \ldots, \mathcal{I}$ in the formal sector, the production function representing the technology is

$$Y_1^i = F_1\left(K_1^i, H^i, G_1\right) = \alpha_1 K_1^i + \alpha_H H^i + A_1 G_1, \quad \alpha_1 > 0,\ \alpha_H > 0,\ A_1 > 0$$

For $j = 1, 2$, the production function representing the technology in the informal sector is given by

$$Y_2^j = F_2\left(K_2^j, L^j, G_2\right) = \alpha_2 K_2^j + \alpha_L L^j + A_2 G_2, \quad \alpha_2 > 0,\ \alpha_L > 0,\ A_2 > 0$$

The government's infrastructural expenditures in the formal and informal sectors are denoted, respectively, by G_1 and G_2. These are taken as fixed/given by the producers. Let price of the consumption good produced in the formal sector be normalised to one. We denote the price of the consumption good produced in the informal sector as p. It follows from the technological specifications that, at all time points t, profit maximisation with respect to human and physical capital in the formal sector and unskilled labour force and physical capital in the informal sector result in the following conditions:

$$r_1(t) = \alpha_1, \quad r_2(t) = \alpha_2 p(t), \quad w_1(t) = \alpha_H, \quad w_2(t) = \alpha_L p(t), \tag{18}$$

[13]Production technologies with constant marginal products are often assumed in the public economics literature. See for example, Atkinson and Stiglitz (1976, 1980), Saez (2001), Mirrlees (1971, 1976), Murty and Ray (1989) among many others. This assumption sharpens focus on specific research questions as it implies that producer prices remain fixed. The tax rates themselves are policy variables and will imply variability in the prices the consumers face. In our analysis in the growth framework too, the above assumption will imply that the interest and wage rates faced by producers are constant. However, the wage and interest rates faced by the consumers will vary depending on the tax rates implemented. Thus, price effects of changes in tax policies will persist throughout the analysis. The extension of the AK model employed in this paper has also been referred to in Chapter 2 of the text book by Acemoglu (2009). The ensuing linear production technology is also employed in the public economics text by Salaniè (2003) to simplify the analysis while deriving the features of the optimal capital tax in an infinite horizon model.

[14]Note that in the pure formal sector case, this is one; while in the most general case this is two.

The profit of each of the formal sector firms in time period t is A_1G_1. Since members of a household who work in the informal sector are self employed, the profit income of any household m from its participation in the informal sector is given by

$$\Pi_{2m}(t) = p(t)A_2G_2 \tag{19}$$

It is clear that such profits generated in the two sectors are sheer economic rents.

5 Budget balance in the government sector and market equilibrium.

To ensure that production efficiency holds at a second-best optimum it is often assumed in the classic public economics literature that profits of the private sector firms are entirely taxed away (*i.e.,* there is 100% corporate profit taxation) whenever technology exhibits decreasing returns to scale.[15] This assumption abstracts from dividend incomes to consumers. To simplify analysis, we too will take this approach and assume that profits of/economic rents accruing to firms in the formal sector are fully taxed.[16]

We assume that the government's expenditure on the public good provided to the consumers is proportional to $\mathcal{C}_1 := \sum_m [C_{1m} + \mu_1]$, *i.e.,*

$$G_c \;=\; g_c\left(\sum_m [C_{1m} + \mu_1]\right) = g_c\mathcal{C}_1, \quad g_c \in (0,1]$$

We will assume that g_c is exogenously fixed. Under these assumptions, the government's budget constraint is:[17]

$$\tau_b r_b B(t) + \tau_k r_1 K_1(t) + \tau_w w_1 H(t) + \tau_c C_1(t) + \mathcal{I}A_1G_1(t) = T(t) + G_c(t) + G_1 + p(t)G_2 \tag{20}$$

[15]See, for example, Guesnerie (1977, 1995) and Murty (2013, 2019). Technology set Y exhibits decreasing returns if whenever a production vector y is in the technology set Y then so is any other production vector of the form κy, where $\kappa \in [0,1]$. That is, whenever scaling down production is always technologically feasible. See, for example, Chapter 5 in Mas-Colell et al (1995). It can be shown that our technology sets in the formal and informal sectors as defined in Section 4 exhibit diminishing returns in physical and human capital in the formal sector and unskilled labour and physical capital in the informal sector, respectively, when government's infrastructural investments in the two sectors are held fixed. Hence, positive profits are generated.

[16]This assumption can also be relaxed so that such profits can be assumed to be partly taxed and partly returned to the households as dividend income.

[17]We denote aggregate consumption of good 1 by $C_1 := \sum_{m=1,2} C_{1m}$. Similarly, we define the aggregate consumption of good 2 by C_2, aggregate bond holding B, aggregate levels of formal sector physical and human capital stocks K_1 and H, respectively.

This says that the total tax revenue from (i) taxing income from holding bonds and physical and human capital in the formal sector, (ii) taxing consumption of good produced in the formal sector, and (iii) hundred percent taxation of profits generated in the formal sector is equal to the total governmental expenditure incurred on the redistributive transfer, provision of public good to the consumers, and infrastructure provision in the formal and informal sectors.

It will be seen in Section 6.7 that G_1 and G_2 are endogenously determined by the model, so that from (21) it follows that the transfer will be given by

$$T(t) = -g_c \mathcal{C}_1(t) - pG_2 + (\mathcal{I}A_1 - 1)G_1 + \tau_b r_b B(t) + \tau_k r_1 K_1(t) + \tau_w w_1 H(t) + \tau_c C_1(t) \quad (21)$$

Under our assumptions, (14) and (18) imply that the price of the good produced in the informal sector is a constant given by :

$$p = \frac{(1-\tau_b)r_b + \delta_2}{\alpha_2} \quad (22)$$

Combined with the assumptions of constant marginal products of inputs in the formal and informal sectors in Section 4 and the profit maximisation conditions in (18), this implies that factor prices faced by the producers are constant:

$$r_1 = \alpha_1, \quad r_2 = \alpha_2 p, \quad w_1 = \alpha_H, \quad w_2 = \alpha_L p, \quad (23)$$

Thus, (22) and (23) imply that there are no transitional dynamics in the decentralised macroeconomic equilibrium for factor prices and price of the informal sector good in this model.

We assume that the formal sector good is a tradable good, while the informal sector good is non-tradable. Further, as seen from the household budget constraint in Section 2.3, investment in all forms of capital and the bond, including physical capital employed in the informal sector, is always done using the numeraire good, *i.e.,* the formal sector good. Output of the informal sector good is not invested for physical capital formation and is only consumed. Hence, in a decentralised equilibrium we will have

$$Y_2 = C_2 \quad (24)$$

Summing up individual instantaneous budget constraints (1) in Section 2.3 and recalling

(19) we obtain[18]

$$\begin{aligned}
\dot{B} &= w_1 H + r_1 K_1 + w_2 L + r_2 K_2 + r_b B \\
&-C_1 - pC_2 - \sum_m X_m \left(1 + \frac{S(\beta_m)X_m}{2L_m}\right) - \sum_m I_{k_1 m}\left(1 + \frac{\pi I_{k_1 m}}{2K_{1m}}\right) \\
&-\sum_m I_{k_{2m}} - \sum_m I_{hm}\left(1 + \frac{\theta I_{hm}}{2Hm}\right) \\
&+T - \tau_w w_1 H - \tau_k r_1 K_1 - \tau_b r_b B - \tau_c C_1 + pA_2 G_2 \sum_m \Lambda_m
\end{aligned}$$

The factor market equilibrium conditions in (23) and the technological specifications in Section 4 imply that $w_2 L + r_2 K_2 = pY_2$. Recalling equilibrium in market for the informal sector good (see (24)) and substituting for T from (21)), the above can be re-written as

$$\begin{aligned}
\dot{B} &= w_1 H + r_1 K_1 + r_b B \\
&-C_1 - \sum_m X_m \left(1 + \frac{S(\beta_m)X_m}{2L_m}\right) - \sum_m I_{k_1 m}\left(1 + \frac{\pi I_{k_1 m}}{2K_{1m}}\right) \\
&-\sum_m I_{k_{2m}} - \sum_m I_{hm}\left(1 + \frac{\theta I_{hm}}{2Hm}\right) \\
&-g_c \mathcal{C}_1(t) - pG_2 + (\mathcal{I}A_1 - 1)G_1
\end{aligned} \tag{25}$$

Recalling (23) and the specification of production technologies in Section 4, the above can be written as

$$\begin{aligned}
\dot{B} &= Y_1 + r_b B - C_1 - \sum_m X_m \left(1 + \frac{S(\beta_m)X_m}{2L_m}\right) - \sum_m I_{k_1 m}\left(1 + \frac{\pi I_{k_1 m}}{2K_{1m}}\right) \\
&-\sum_m I_{k_{2m}} - \sum_m I_{hm}\left(1 + \frac{\theta I_{hm}}{2Hm}\right) \\
&-g_c \mathcal{C}_1(t) - pG_2 - G_1
\end{aligned} \tag{26}$$

Thus, if all consumers satisfy their instantaneous budget constraints, the government's budget constraint is satisfied, and market for the informal sector good clears, then so does the market for the formal sector good in an open economy setting.This is the implication for a general equilibrium that directly follows from the Walras law. Clearly, (26) indicates that, in equilibrium, the total supply of good 1 at any time point, which is the total production of the formal sector good Y_1 and the income earned on bond holdings $r_b B$ in the economy can be con-

[18] Note that the aggregate bond holding is defined as $B = \sum_m B_m$. Similarly, we can define aggregate levels of physical and human capital, consumption, and the net unskilled labour force in the informal sector: K_1, K_2, H, C_1, C_2, and L.

sumed, invested in human and physical capital with adjustment costs, and used for educating the migrating labour force, for producing government infrastructure in production, for financing production of a public good for consumption, and for acquisition of foreign bonds. Furthermore, (26) allows for cases where use of this good for domestic consumption and investment can be less or more than the domestic production allowing for the possibilities of exportation or importation this good in line with our assumption that this good is tradable.

Hence, for specifying a market equilibrium in every time period, it suffices to specify equilibrium in the market for the good produced in the informal sector market (24) and that the government budget is balanced (21). These automatically imply (26), *i.e.,* the market for good 1 also clears.

6 Solving for a decentralised macro-economic equilibrium of the model.

6.1 Decentralised growth paths of consumption.

Recalling from Section 2.2 that function c is assumed to have a Stone-Geary structure, we define $\mathcal{C}_{1m} = C_{1m} + \mu_1$, $\mathcal{C}_{2m} = C_{2m} - \mu_2$, and function $\mathcal{U}$ as

$$\mathcal{U}\left(\mathcal{C}_{1m}, \mathcal{C}_{2m}, G_c\right) = \frac{1}{1-\epsilon}\left(\mathcal{C}_{1m}^{\eta_1}\mathcal{C}_{2m}^{\eta_1}G_c\right)^{1-\epsilon},$$

Then $U\left(C_{1m}, C_{2m}, G_c\right) = \mathcal{U}\left(\left(C_{1m}+\mu_1\right), \left(C_{2m}-\mu_2\right), G_c\right)$. Condition (3) of intertemporal consumer optimisation yields

$$\mathcal{C}_{2m} = \chi_1 \mathcal{C}_{1m} \tag{27}$$

where $\chi_1 = \frac{\eta_2}{\eta_1}\frac{(1+\tau_c)}{p}$. Hence, we have

$$c_m = \left(C_{1m}+\mu_1\right)^{\eta_1}\left(C_{2m}-\mu_2\right)^{\eta_2} = \chi_1^{\eta_2}\mathcal{C}_{1m} \tag{28}$$

Now going back to the first order conditions in (2), we obtain

$$\begin{aligned} U_1 = \frac{\partial U}{\partial C_{1m}} = \mathcal{U}_1 = \frac{\partial \mathcal{U}_1}{\partial \mathcal{C}_{1m}} &= \lambda_{bm}(1+\tau_c) \\ \Longrightarrow \chi \mathcal{C}_{1m}^{-\epsilon} G_c^{1-\epsilon} &= \lambda_{bm}(1+\tau_c) \end{aligned}$$

where $\chi = \eta_1 \chi_1^{\eta_2(1-\epsilon)} = \left[\frac{\eta_2(1+\tau_c)}{p}\right]^{\eta_2(1-\epsilon)} \eta_1^{(1-\eta_2(1-\epsilon))}$.

Differentiating both sides with respect to t and dividing by $\mathcal{U}_1$ and employing (9), we obtain

$$\frac{\dot{\mathcal{C}}_{1m}(t)}{\mathcal{C}_{1m}(t)} = \frac{(1-\tau_b)r_b - \rho}{\epsilon} =: \psi_{\mathcal{C}}, \tag{29}$$

Hence we have

$$\begin{aligned} \mathcal{C}_{1m}(t) &= \mathcal{C}_{1m}(0)e^{\frac{[(1-\tau_b)r_b-\rho]t}{\epsilon}} = \mathcal{C}_{1m}(0)e^{\psi_{\mathcal{C}}t} \\ \implies C_{1m}(t) &= \left(C_{1m}(0) + \mu_1\right)e^{\psi_{\mathcal{C}}t} - \mu_1 \end{aligned} \tag{30}$$

From (27) and (30) we have

$$\begin{aligned} \mathcal{C}_{2m} &= \frac{\eta_2}{\eta_1}\frac{(1+\tau_c)}{p}\mathcal{C}_{1m}(0)e^{\psi_{\mathcal{C}}t} \\ \implies C_{2m}(t) &= \frac{\eta_2}{\eta_1}\frac{(1+\tau_c)}{p}\mathcal{C}_{1m}(0)e^{\psi_{\mathcal{C}}t} + \mu_2 \end{aligned} \tag{31}$$

Thus, at solutions to households' intertemporal welfare maximisation, $\mathcal{C}_{1m}$ and $\mathcal{C}_{2m}$ have the same rate of growth $\psi_{\mathcal{C}}$. However, C_{1m} and C_{2m} grow at different rates in the short-run. But, in the long-run, their rates of growth are the same and equal to $\psi_{\mathcal{C}}$.

6.2 Equilibrium values of variables with no transitional dynamics.

We will construct a decentralised macroeconomic tax equilibrium where[19]

$$\frac{\dot{\lambda}_{k_1bm}}{\lambda_{k_1bm}} = \frac{\dot{\lambda}_{hbm}}{\lambda_{hbm}} = \frac{\dot{\lambda}_{lbm}}{\lambda_{lbm}} = 0. \tag{32}$$

Thus, the relative shadow prices of physical and human capital in the formal sector and the shadow price of the unskilled labour λ_{k_1bm}, λ_{hbm}, and λ_{lbm}, respectively, do not display any transitional dynamics at such an equilibrium. At these values of the shadow prices, equations (11), (12), (13) imply[20]

$$\frac{(1-\tau_k)r_1}{\lambda_{k_1bm}} + \frac{(\lambda_{k_1bm}-1)^2}{2\pi\lambda_{k_1bm}} - \delta_{k1} = (1-\tau_b)r_b \tag{33}$$

[19]This is similar to the methodology followed by Turnovsky (1996).

[20]In this section, we will derive the equilibrium values of these relative shadow prices.

$$\frac{(1-\tau_w)w_1}{\lambda_{hbm}} + \frac{(\lambda_{hbm}-1)^2}{2\theta\lambda_{hbm}} = (1-\tau_b)r_b \tag{34}$$

$$\frac{w_2 + S(\beta_m)\frac{X_m{}^2}{2L_m{}^2}}{\lambda_{lbm}} = (1-\tau_b)r_b \tag{35}$$

Additionally, we also re-write (4) and (5) as

$$\lambda_{hbm}\beta_m - \lambda_{lbm} = 1 + \frac{S(\beta_m)\hat{X}_m}{\hat{L}_m} \tag{36}$$

$$X_m\left[\lambda_{hbm} - \frac{\hat{X}_m}{2\hat{L}_m}S'(\beta_m)\right] = 0 \tag{37}$$

Define net-of-migration unskilled labour force in household m per unit total unskilled labour force born in household m as $\hat{L}_m = \frac{L_m}{l_m}$ and the migration per unit total unskilled labour force born in household m as $\hat{X}_m = \frac{X_m}{l_m}$. Then (17) becomes

$$\dot{\hat{L}}_m = n - \hat{X}_m - n\hat{L}_m \tag{38}$$

Similarly, defining $\hat{H}_m := \frac{H_m}{l_m}$, (16) becomes

$$\dot{\hat{H}}_m = \left(\frac{\lambda_{hbm} - n\theta - 1}{\theta}\right)\hat{H}_m + \beta_m\hat{X}_m \tag{39}$$

6.2.1 Solving for decentralised equilibrium values of λ_{hb}, λ_{lb}, $\frac{\hat{X}}{\hat{L}}$ and β.

We obtain the growth paths of $\hat{X}_m$ and β_m and the steady state values of λ_{hbm} and λ_{lbm} as follows by solving equations (34) to (37).

Since we are looking for interior solutions to the intertemporal utility maximisation by household, we ignore the case $X_m = 0$. Equation (37) then implies that $\lambda_{hbm} - \frac{X_m}{2L_m}S'(\beta_m) = 0$, from whence it follows that

$$\lambda_{hbm} - \frac{\hat{X}_m}{2\hat{L}_m}S'(\beta_m) = 0 \tag{40}$$

From equation (34), we obtain the steady state values of λ_{hbm} by solving the following quadratic equation:

$$\frac{1}{2\theta}\lambda_{hbm}^2 - (\frac{1}{\theta} + (1-\tau_b)r_b)\lambda_{hbm} + \left(\frac{1}{2\theta} + (1-\tau_w)w_1\right) = 0 \tag{41}$$

This equation has real roots if and only if

$$\begin{aligned}\left(\frac{1}{\theta} + (1-\tau_b)r_b\right)^2 &- \left(\frac{1}{\theta^2} + \frac{2(1-\tau_w)w_1}{\theta}\right) \geq 0 \\ &\iff 2(1-\tau_b)r_b\theta + (1-\tau_b)^2 r_b^2\theta^2 - 2\theta(1-\tau_w)w_1 \geq 0.\end{aligned} \tag{42}$$

In that case, the two roots of equation (34) are obtained as

$$\begin{aligned}\lambda_{hbm} &= \left(1 + (1-\tau_b)r_b\theta\right) \pm \theta\sqrt{\left(\frac{1}{\theta} + (1-\tau_b)r_b\right)^2 - 4.\frac{1}{2\theta.}\left(\frac{1}{2\theta} + (1-\tau_w)w_1\right)} \\ &= \left(1 + (1-\tau_b)r_b\theta\right) \pm \sqrt{(1-\tau_b)^2 r_b^2\theta^2 + 2\theta\left[(1-\tau_b)r_b - (1-\tau_w)w_1\right]}\end{aligned} \tag{43}$$

Let λ_{hbm}^1 and λ_{hbm}^2 denote the negative and positive roots in (43). Hence, $\lambda_{hbm}^1 \leq \lambda_{hbm}^2$. Assuming that the roots are distinct, two cases arise, which are:

$$\begin{aligned}&\text{(i)} \quad (1-\tau_w)w_1 > (1-\tau_b)r_b \implies 1 < \lambda_{hbm}^1 < \lambda_{hbm}^2 \\ &\text{(ii)} \quad (1-\tau_w)w_1 < (1-\tau_b)r_b \implies 0 < \lambda_{hbm}^1 < 1 < \lambda_{hbm}^2\end{aligned} \tag{44}$$

Using equation (35), we obtain

$$\lambda_{lbm} = \frac{w_2 + S(\beta_m)\frac{\hat{X}_m^2}{2\hat{L}_m^2}}{(1-\tau_b)r_b}. \tag{45}$$

Using equation (40), we obtain the value of $\frac{\hat{X}_m}{\hat{L}_m}$.

$$\frac{\hat{X}_m}{\hat{L}_m} = \frac{2\lambda_{hbm}}{S'(\beta_m)}. \tag{46}$$

Substituting (45) and (46) for λ_{lbm} and $\frac{\hat{X}_m}{\hat{L}_m}$ in equation (36) we obtain β_m as an implicit function of λ_{hbm} and other parameters as follows:

$$\lambda_{hbm}[\beta_m - \frac{2S(\beta_m)}{S'(\beta_m)}] = 1 + \lambda_{lbm} \tag{47}$$

$$\implies \frac{2S(\beta_m)}{S'(\beta_m)^2}\lambda_{hbm}^2 - \left[\frac{\beta_m S'(\beta_m) - 2S(\beta_m)}{S'(\beta_m)}\right](1-\tau_b)r_b\lambda_{hbm} + \left((1-\tau_b)r_b + w_2\right) = 0 \tag{48}$$

Thus, using the values of λ_{hbm} derived in (43), equation (48) can be employed to solve for the equilibrium value of β_m as functions of the parameters of the household's intertemporal utility maximisation problem.

Employing the values of β_m and λ_{hbm}, we can derive the equilibrium value of $\frac{\hat{X}_m}{\hat{L}_m}$ using (46), and then we can derive the equilibrium value of λ_{lbm} using (45).

Note from (43) and (48) that the computed equilibrium values of λ_{hbm}, and β_m depend only on the parameter values of intertemporal utility maximisation (excluding the initial endowments). Hence, they are the same for all consumers. Thus, at an equilibrium, we can write $\lambda_{hbm} = \lambda_{hb}$ and $\beta_m = \beta$ for all m. This implies from (45) that λ_{lbm} is also a constant across consumers $m = 1, 2$. Let's denote it by say $\lambda_{lbm} = \lambda_{lb}$. This further implies from (46) that $\frac{\hat{X}_m}{\hat{L}_m}$ is also a constant given by

$$\frac{\hat{X}_m}{\hat{L}_m} = \frac{2\lambda_{hb}}{S'(\beta)} = \kappa \ \forall \ m, \tag{49}$$

where κ is a function of the parameters of the utility maximisation problem (excluding the initial endowments) that are common across all consumers. Hence, κ is migration of unskilled labour in the informal sector as a proportion of the net labour force employed in this sector.

Thus, in the macroeconomic tax equilibrium that we will construct, which is characterised by (32), there are no transitional dynamics for λ_{hb}, λ_{lb}, β, and κ.

6.2.2 Restrictions on function S.

Note that (47) implies

$$\frac{\lambda_{hb}}{1 + \lambda_{lb}} = \frac{S'(\beta)}{S'(\beta)\beta_m - 2S(\beta)} \tag{50}$$

Since $\lambda_{hb} > 0$ and $\lambda_{lb} > 0$, the above implies the following restriction on function S:

$$\frac{S'(\beta)}{S'(\beta)\beta - 2S(\beta)} > 0$$
$$\implies S'(\beta)\beta > 2S(\beta)$$

An example of a functional form that satisfy this condition is

$$S(\beta) = \upsilon\beta^q, \text{ for } q > 2 \text{ and } \upsilon > 0.$$

6.3 Decentralised growth paths of L_m and X_m.

Equation (49) implies that the rate of growth of $\hat{X}_m$ is the same as the rate of growth of $\hat{L}_m$ and $\hat{X}_m = \kappa \hat{L}_m$. Replacing $\hat{X}_m$ by $\kappa \hat{L}_m$ in equation (38) we obtain the differential equation

$$\dot{\hat{L}}_m + (n+\kappa)\hat{L}_m = n,$$

solving which we obtain the growth path $\hat{L}_m(t)$ as

$$\hat{L}_m(t) = \left(\hat{L}_m(0) - \frac{n}{n+\kappa}\right) e^{-(n+\kappa)t} + \frac{n}{n+\kappa} \tag{51}$$

This implies that

$$L_m(t) \quad = \quad \mathcal{E}_m e^{-\kappa t} + \mathcal{F}_m e^{nt} \tag{52}$$

where $\mathcal{E}_m = L_m(0) - l_m(0)\frac{n}{n+\kappa}$ and $\mathcal{F}_m = l_m(0)\frac{n}{n+\kappa}$. From this it is clear that the long-run rate of growth of $L_m(t)$ is given by

$$\lim_{t\to\infty} \frac{\dot{L}_m(t)}{L_m(t)} = n \qquad \forall\ m \tag{53}$$

Since $\hat{X}_m(t) = \kappa \hat{L}_m(t)$ we can use (56) to also derive the growth path of migration X as follows

$$X_m(t) \quad = \quad \kappa L_m(t) = \kappa\left[L_m(0)e^{-\kappa t} - l_m(0)\frac{n}{n+\kappa}e^{-\kappa t} + \frac{n}{n+\kappa}l_m(0)e^{nt}\right] \tag{54}$$

The transversality condition for unskilled labour in the informal sector is

$$\lim_{t\to\infty} \lambda_{lm}(t)L_m(t)e^{-\rho t} = 0.$$

Recalling that $\lambda_{lm}(t) = \lambda_{lb}\lambda_{bm}(t)$, this can be re-written in terms of $\hat{L}$ as

$$\lim_{t\to\infty} \lambda_{lb}\lambda_{bm}(t)l(t)\hat{L}_m(t)e^{-\rho t} = 0,$$

$$\implies l_m(0)\lambda_{lb}\lambda_{bm}(0) \lim_{t\to\infty} e^{(n-(1-\tau_b)r_b)t}\hat{L}_m(t) = 0 \tag{55}$$

where λ_{lb} is given by (45) and $\lambda_{bm}(t)$ is given by (10). From (51) it follows that

$$\lim_{t\to\infty} \hat{L}_m(t) = \frac{n}{n+k}.$$

Hence, the transversality condition for L_m in equation (55) is true if and only if

$$n - (1-\tau_b)r_b < 0 \iff n < (1-\tau_b)r_b \tag{56}$$

i.e., if and only of the population rate of growth is less than the after tax rate of return on the bond.

6.4 Decentralised growth path of H_m.

From (39), (51), and the fact that in equilibrium $\hat{X}_m(t) = \kappa \hat{L}_m(t)$, we obtain

$$\dot{\hat{H}}_m - \gamma \hat{H}_m = \beta\kappa \left[\left(\hat{L}_m(0) - \frac{n}{n+\kappa} \right) e^{-(n+\kappa)t} + \frac{n}{n+\kappa} \right], \tag{57}$$

where β solves (48) and we define

$$\gamma \;:=\; \frac{\lambda_{hb} - n\theta - 1}{\theta} \;=\; \frac{\lambda_{hb} - 1}{\theta} - n \tag{58}$$

Thus, γ is the rate of investment in human capital less the growth in the gross unskilled labour force. Solving this differential equation, we obtain

$$\hat{H}_m(t) = \hat{H}_m(0)e^{\gamma t} - \frac{\beta\kappa}{n+\kappa+\gamma} \left(\hat{L}_m(0) - \frac{n}{n+\kappa} \right) \left[e^{-(n+\kappa)t} - e^{\gamma t} \right] - \frac{\beta\kappa n}{(n+\kappa)\gamma} \left[1 - e^{\gamma t} \right] \tag{59}$$

The path of $H_m(t)$ is then obtained as:

$$\begin{aligned} H_m(t) \;&=\; \left[H_m(0) + \frac{\beta\kappa}{n+\kappa+\gamma} \left(L_m(0) - l_m(0)\frac{n}{n+\kappa} \right) + l_m(0)\frac{\beta\kappa n}{(n+\kappa)\gamma} \right] e^{(\gamma+n)t} \\ &\quad - \frac{\beta\kappa n}{(n+\kappa)\gamma} l_m(0) \left(1 - \frac{\gamma}{n+\kappa+\gamma} \right) e^{nt} - L_m(0)\frac{\beta\kappa}{n+\kappa+\gamma} e^{-\kappa t} \\ &\implies H_m(t) = \mathcal{A}_m e^{(n+\gamma)t} - \mathcal{B}_m e^{nt} - \mathcal{D}_m e^{-\kappa t} \end{aligned} \tag{60}$$

where $\mathcal{A}_m$, $\mathcal{B}_m$, and $\mathcal{D}_m$ are defined in an obvious way.

6.4.1 The transversality condition for H_m.

The transversality condition for human capital in the formal sector is

$$\lim_{t\to\infty} \lambda_{hm}(t)H_m(t)e^{-\rho t} = 0.$$

This can be re-written in terms of $\hat{H}$ as

$$l_m(0)\lambda_{hb}\lambda_{bm}(0) \lim_{t\to\infty} e^{(n-(1-\tau_b)r_b)t}\hat{H}_m(t) = 0 \tag{61}$$

where where λ_{hb} is given by (43). Given the growth path of $\hat{H}_m$ specified in (59) and the fact that $\kappa > 0$, the transversality condition will be true if and only if

$$\gamma + n - (1-\tau_b)r_b < 0$$

i.e., if and only if the rate of investment in human capital is less than the after-tax return on bond.

The condition above is always true for the negative root of λ_{hb} in (43) as then

$$\gamma+n-(1-\tau_b)r_b = \frac{\lambda_{hb} - 1 - (1-\tau_b)r_b\theta}{\theta} = -\frac{\sqrt{\theta\left(2(1-\tau_b)r_b + (1-\tau_b)^2 r_b^2\theta - 2(1-\tau_w)w_1\right)}}{\theta} < 0.$$

On the other hand if we consider the positive root of λ_{hb} in (43), then

$$\gamma + n - (1-\tau_b)r_b = \frac{\sqrt{\theta\left(2(1-\tau_b)r_b + (1-\tau_b)^2 r_b^2\theta - 2(1-\tau_w)w_1\right)}}{\theta} > 0.$$

Hence, satisfaction of the transversality condition requires choosing λ_{hb} as the negative root in (43) . But (12) implies that

$$\dot{\lambda}_{hb} = (1-\tau_b)r_b\lambda_{hb} - (1-\tau_w)w_1 - \frac{(\lambda_{hb}-1)^2}{2\theta},$$

which implies that

$$\frac{\partial\dot{\lambda}_{hbm}}{\partial\lambda_{hb}} = (1-\tau_b)r_b - \frac{(\lambda_{hb}-1)}{\theta} =: \xi,$$

Stability will hence require $\xi < 0$, which will be true if and only if the positive root of λ_{hb} in (43) is chosen.

Hence, although the negative root of λ_{hb} in (43) leads to an unstable steady state for λ_{hb}, it is the one that ensures that the transversality condition for human capital H is satisfied at the consumer optimum. So we will proceed with the analysis taking the negative root of λ_{hb}. This implies that λ_{hb} will jump to its equilibrium value in time period zero and will not change over time.

6.4.2 Long-run growth rate of H_m.

Employing (60) it can be shown that

$$\begin{aligned}\frac{\dot{H}_m(t)}{H_m(t)} &= \frac{n+\gamma}{1-e^{-\gamma t}\mathcal{B}_m/\mathcal{A}_m - e^{-(n+\gamma+\kappa)t}\mathcal{D}_m/\mathcal{A}_m} - \frac{n}{e^{\gamma t}\mathcal{A}_m/\mathcal{B}_m - 1 - e^{-(n+\kappa)t}\mathcal{D}_m/\mathcal{B}_m} \\ &+ \frac{\kappa}{e^{(n+\gamma+\kappa)t}\mathcal{A}_m/\mathcal{D}_m - e^{(n+\kappa)t}\mathcal{B}_m/\mathcal{D}_m - 1} \\ \Longrightarrow \lim_{t\longrightarrow\infty}\frac{\dot{H}_m(t)}{H_m(t)} &= n+\gamma \text{ if } \gamma>0 \quad \text{and} \quad \lim_{t\longrightarrow\infty}\frac{\dot{H}_m(t)}{H_m(t)} = n \text{ if } \gamma\leq 0 \qquad \forall\, m \end{aligned} \tag{62}$$

Hence, the long-run growth rate of human capital is $\max\{n, n+\gamma\}$. In particular, we have

$$\lim_{t\longrightarrow\infty}\frac{\dot{H}_m(t)}{H_m(t)} = n+\gamma = \frac{\lambda_{hb}-1}{\theta} \geq 0 \quad \text{if } \gamma>0.$$

Given the fact that the negative root of λ_{hb} satisfies the transversality condition for human capital, (44) implies that the above will be true if and only if $(1-\tau_w)w_1 > (1-\tau_b)r_b$, *i.e.*, the after-tax rate of return on human capital is bigger than the after-tax rate of return on the bond.

6.5 Decentralised growth path of K_{1m}.

Equation (15) implies that

$$K_{1m}(t) = K_{1m}(0)e^{\psi_{k_1 m}(t)}.$$

It follows from the definition of $\psi_{k_1 m}$ in (15) and (32) that K_{1m} grows at a constant rate in this economy. Solving (33) for $\lambda_{k_1 bm}$, we obtain its two potential steady state values:

$$\lambda_{k_1 bm} = 1+\pi\delta_{k_1}+\pi(1-\tau_b)r_b \pm \sqrt{(1+\pi\delta_{k_1}+\pi(1-\tau_b)r_b)^2 - (1+2\pi r_1(1-\tau_k))} \tag{63}$$

Hence, (15) implies that the possible long-run growth rates of K_{1m} are

$$\begin{aligned}\psi_{k_1m} &= \frac{\pi(1-\tau_b)r_b \pm \sqrt{(1+\pi\delta_{k_1}+\pi(1-\tau_b)r_b)^2-(1+2\pi r_1(1-\tau_k))}}{\pi} \\ &= \frac{\pi(1-\tau_b)r_b \pm \sqrt{(\pi\delta_{k_1}+\pi(1-\tau_b)r_b)^2+2\left(\pi\delta_{k_1}+\pi(1-\tau_b)r_b\right)-2\pi r_1(1-\tau_k)}}{\pi} \quad (64)\end{aligned}$$

Once again note that the steady state values of λ_{k_1bm} and the implied growth rates of K_{1m}, *i.e.*, the two possible values of ψ_{k_1m}, are independent of m. Hence, we can write $\lambda_{k_1bm} = \lambda_{k_1b}$ and $\psi_{k_1m} = \psi_{k_1}$. The transversality condition for physical capital in the formal sector is

$$\lim_{t\to\infty} \lambda_{k_1m}(t)K_{1m}(t)e^{-\rho t} = 0.$$

This implies

$$K_{1m}(0)\lambda_{k_1b}\lambda_{bm}(0)\lim_{t\to\infty} e^{\left(\psi_{k_1}-(1-\tau_b)r_b\right)t} = 0 \quad (65)$$

where values of λ_{k_1b} are given by (63). Thus, the transversality condition for K_{1m} will hold if and only if

$$\psi_{k_1} < (1-\tau_b)r_b, \quad (66)$$

i.e., if and only if the rate of growth of physical capital in the formal sector is less than the after-tax rate of return on the bond. Given the definition of ψ_{k_1} in (15), equations (63) and (64) indicate that the above will be true if and only if λ_{k_1b} is chosen as the negative root in (63). Further more, given this choice of λ_{k_1b}, a necessary but not sufficient condition for a positive rate of growth of physical capital in the formal sector, *i.e.*, $\psi_{k_1} > 0$, is

$$\psi_{k_1} > 0 \implies (1-\tau_{k_1})r_1 - \delta_{k_1} > (1-\tau_b)r_b. \quad (67)$$

6.6 Equilibrium growth paths of K_{2m}, I_{k_2m}, and initial level of aggregate household consumption.

At any period t, recalling the technology in the informal sector, $K_2(t)$ is obtained from the market equilibrium condition for good two (24) and the expression for C_2 in Section 6.1 as follows:

$$\begin{aligned} Y_2 = C_2 &= \sum_m \chi_1 \mathcal{C}_{1m} + 2\mu_2 \\ \implies \alpha_2 K_2(t) + \alpha_L L(t) + A_2 G_2 \sum_m \Lambda_m &= \frac{\eta_2}{\eta_1} \frac{(1+\tau_c)}{p} \sum_m \mathcal{C}_{1m}(t) + 2\mu_2 \end{aligned} \tag{68}$$

This implies that

$$K_2(t) = \mathcal{K}_{\mathcal{C}} \mathcal{C}_1(t) - \mathcal{K}_{\mathcal{L}} L(t) + 2\mathcal{K}_{\mathcal{M}} - \frac{A_2 G_2 \sum_m \Lambda_m}{\alpha_2} \tag{69}$$

where

$$\mathcal{C}_1(t) = \mathcal{C}_{11}(t) + \mathcal{C}_{12}(t), \quad \mathcal{K}_{\mathcal{C}} = \frac{\eta_2(1+\tau_c)}{\eta_1 p \alpha_2}, \quad \mathcal{K}_{\mathcal{L}} = \frac{\alpha_L}{\alpha_2}, \quad \text{and} \quad \mathcal{K}_{\mathcal{M}} = \frac{\mu_2}{\alpha_2} \tag{70}$$

Further, since the shadow prices of K_2 and B are the same (see (7)), people are indifferent between investing in bonds B and investing in physical capital in the informal sector K_2. Hence, there are potentially many solutions for bond accumulation $\dot{B}_m$ and investment in informal-sector physical capital $I_{k_2 m}$ in the intertemporal utility maximization problem of the household m in Section 3. Let us assume that each household accumulates the informal sector physical capital at the same rate $\psi_{k_2}(t)$ according to the rule

$$K_{2m}(t) = K_{2m}(0) e^{\psi_{k_2}(t)} \tag{71}$$

Since $\sum_m K_{2m} = K_2$, from (69) it follows that

$$\sum_m K_{2m}(t) = K_2(0) e^{\psi_{k_2}(t)} = \mathcal{K}_{\mathcal{C}} \mathcal{C}_1(t) - \mathcal{K}_{\mathcal{L}} L(t) + 2\mathcal{K}_{\mathcal{M}} - \frac{A_2 G_2 \sum_m \Lambda_m}{\alpha_2} = K_2(t)$$

Solving the above for $\psi_{k_2}(t)$, we obtain the growth rate of K_2 as

$$\psi_{k_2}(t) = \ln \left(\frac{\mathcal{K}_{\mathcal{C}} \mathcal{C}_1(t) - \mathcal{K}_{\mathcal{L}} L(t) + 2\mathcal{K}_{\mathcal{M}}}{K_2(0)} - \frac{A_2 G_2 \sum_m \Lambda_m}{\alpha_2 K_2(0)} \right) \tag{72}$$

From (71) and (72) it follows that

$$K_{2m}(t) = k_{2m}(0) \left(\mathcal{K}_{\mathcal{C}} \mathcal{C}_1(t) - \mathcal{K}_{\mathcal{L}} L(t) + 2\mathcal{K}_{\mathcal{M}} - \frac{A_2 G_2 \sum_m \Lambda_m}{\alpha_2} \right) \tag{73}$$

where $k_{2m}(0) = \frac{K_{2m}(0)}{K_2(0)}$ is the initial share of household $m = 1, 2$ in the aggregate informal sector's capital. Thus, $\sum_m k_{2m}(0) = 1$. Now I_{k_2m} can be derived for any household m by employing the fact that

$$I_{k_2m} = \dot{K}_{2m} + \delta_2 K_{2m},$$

where $\dot{K}_{2m}$ is obtained by taking time derivative of (73). Thus,

$$\dot{K}_{2m}(t) = k_{2m}(0)\left[\mathcal{K}_{\mathcal{C}}\mathcal{C}_1(0)\psi_{\mathcal{C}}e^{\psi_{\mathcal{C}}t} + \mathcal{K}_{\mathcal{L}}(L(0) - l(0)\frac{n}{n+\kappa})\kappa e^{-\kappa t} - \mathcal{K}_{\mathcal{L}}l(0)\frac{n}{n+\kappa}ne^{nt}\right] \quad (74)$$

Hence, the investment in the informal sector's capital by household m is directly dependent on $k_{2m}(0)$ and is obtained by using (73) and (74) as:

$$\begin{aligned} I_{k_2m}(t) &= k_{2m}(0)\Big[\mathcal{K}_{\mathcal{C}}\mathcal{C}_1(0)(\psi_{\mathcal{C}} + \delta_2)e^{\psi_{\mathcal{C}}t} + \mathcal{K}_{\mathcal{L}}\left(L(0) - l(0)\frac{n}{n+\kappa}\right)(\kappa - \delta_2)e^{-\kappa t} \\ &\quad -\mathcal{K}_{\mathcal{L}}l(0)\frac{n}{n+\kappa}(n+\delta_2)e^{nt}\Big] \end{aligned} \quad (75)$$

So the aggregate investment in informal sector's capital is:

$$\begin{aligned} I_{k_2}(t) &= \mathcal{K}_{\mathcal{C}}\mathcal{C}_1(0)(\psi_{\mathcal{C}} + \delta_2)e^{\psi_{\mathcal{C}}t} + \mathcal{K}_{\mathcal{L}}\left(L(0) - l(0)\frac{n}{n+\kappa}\right)(\kappa - \delta_2)e^{-\kappa t} \\ &\quad -\mathcal{K}_{\mathcal{L}}l(0)\frac{n}{n+\kappa}(n+\delta_2)e^{nt} \end{aligned} \quad (76)$$

From equation (69) the aggregate stock of physical capital in the informal sector is obtained as

$$K_2(t) = \mathcal{K}_{\mathcal{C}}\mathcal{C}_1(t) - \mathcal{K}_{\mathcal{L}}L(t) + 2\mathcal{K}_{\mathcal{M}} - \frac{A_2 G_2 \sum_m \Lambda_m}{\alpha_2},$$

which implies

$$K_2(0) = \mathcal{K}_{\mathcal{C}}\mathcal{C}_1(0) - \mathcal{K}_{\mathcal{L}}L(0) + 2\mathcal{K}_{\mathcal{M}} - \frac{A_2 G_2 \sum_m \Lambda_m}{\alpha_2}$$

This allows us to obtain the equilibrium initial value of aggregate $\mathcal{C}_1$ and consumption of good one as:

$$\begin{aligned} \mathcal{C}_1(0) &= [K_2(0) + \mathcal{K}_{\mathcal{L}}L(0) - 2\mathcal{K}_{\mathcal{M}} + \frac{A_2 G_2 \sum_m \Lambda_m}{\alpha_2}]\mathcal{K}_{\mathcal{C}}^{-1} \\ \implies C_1(0) &= [K_2(0) + \mathcal{K}_{\mathcal{L}}L(0) - 2\mathcal{K}_{\mathcal{M}} + \frac{A_2 G_2 \sum_m \Lambda_m}{\alpha_2}]\mathcal{K}_{\mathcal{C}}^{-1} - 2\mu_1 \end{aligned} \quad (77)$$

6.7 Decentralised growth paths of B_m and equilibrium values of G_1 and G_2.

The decentralised growth paths of individual and aggregate bond accumulation are derived below. Imposing the transversality conditions on these trajectories yields conditions that endogenously fix the values of government infrastructure expenditures in the two sectors, G_1 and G_2.

6.7.1 Equilibrium aggregate bond accumulation.

Recalling (25) and substituting for the equilibrium growth paths of H using (60), K_1 using (15), X_m using (54), L_m using (52), $\frac{I_{k_1m}}{K_{1m}}$ using (6), I_{k_2} using (76), and $\frac{I_{hm}}{H_m}$ using (8), and $C_1(0)$ and $\mathcal{C}_1(0)$ using (77), the equation for aggregate bond accumulation is given by

$$\begin{aligned}
\dot{B}(t) &= r_b B(t) \\
&+\left\{w_1 - \frac{\lambda^2{}_{hb}-1}{2\theta}\right\}\mathcal{A}e^{(n+\gamma)t} \\
&-\left[\left\{w_1 - \frac{\lambda^2{}_{hb}-1}{2\theta}\right\}\mathcal{B} + \frac{2\lambda_{hb}}{S'(\beta)}\left(1+\frac{S(\beta)\lambda_{hb}}{S'(\beta)}\right)\mathcal{F} - \mathcal{K}_{\mathcal{L}}(n+\delta_2)\mathcal{F}\right]e^{nt} \\
&-\left[\left\{w_1 - \frac{\lambda^2{}_{hb}-1}{2\theta}\right\}\mathcal{D} + \frac{2\lambda_{hb}}{S'(\beta)}\left(1+\frac{S(\beta)\lambda_{hb}}{S'(\beta)}\right)\mathcal{E} + \mathcal{K}_{\mathcal{L}}(\kappa-\delta_2)\mathcal{E}\right]e^{-\kappa t} \\
&+\left[r_1 - \frac{\lambda^2{}_{k_1b}-1}{2\pi}\right]K_1(0)e^{\psi_{k_1}t} - \mathcal{K}_{\mathcal{C}}\mathcal{C}_1(0)(\psi_{\mathcal{C}}+\delta_2)e^{\psi_{\mathcal{C}}t} \\
&-\mathcal{C}_1(0)e^{\psi_{\mathcal{C}}t} + 2\mu_1 + pA_2G_2\sum_m \Lambda_m + \left\{G_1(\mathcal{I}A_1-1) - pG_2 - \mathcal{C}_1(0)g_c e^{\psi_{\mathcal{C}}t}\right\}
\end{aligned}$$

where using the definitions of $\mathcal{A}_m$, $\mathcal{B}_m$, $\mathcal{D}_m$, $\mathcal{E}_m$, and $\mathcal{F}_m$ we define

$$\begin{aligned}
\sum_m \mathcal{A}_m &= H(0) + \frac{\beta\kappa}{n+\kappa+\gamma}\left(L(0) - l(0)\frac{n}{n+\kappa}\right) + l(0)\frac{\beta\kappa n}{(n+\kappa)\gamma} \\
&= H(0) + \frac{\beta\kappa}{n+\kappa+\gamma}L(0) - l(0)\left(\frac{\beta\kappa n}{(n+\kappa)(n+\kappa+\gamma)} - \frac{\beta\kappa n}{(n+\kappa)\gamma}\right) = \mathcal{A}, \\
\sum_m \mathcal{B}_m &= \frac{\beta\kappa n}{(n+\kappa)\gamma}l(0)\left(1 - \frac{\gamma}{n+\kappa+\gamma}\right) = \mathcal{B}, \qquad \sum_m \mathcal{D}_m = L(0)\frac{\beta\kappa}{n+\kappa+\gamma} = \mathcal{D} \\
\sum_m \mathcal{E}_m &= L(0) - l(0)\frac{n}{n+\kappa} = \mathcal{E}, \qquad \sum_m \mathcal{F}_m = l(0)\frac{n}{n+\kappa} = \mathcal{F}
\end{aligned}$$

Solving the differential equation for B, we will obtain

$$\begin{aligned} B(t) &= \Theta e^{r_b t} + \frac{\Gamma_1}{n+\gamma-r_b} e^{(n+\gamma)t} - \frac{\Gamma_2}{n-r_b} e^{nt} + \frac{\Gamma_3}{\kappa+r_b} e^{-\kappa t} + \frac{\Gamma_4}{\psi_{k_1}-r_b} e^{\psi_{k_1} t} \\ & -\frac{\mathcal{C}_1(0)\,(1+\Gamma_5)}{\psi_{\mathcal{C}}-r_b} e^{\psi_{\mathcal{C}} t} - \left(\frac{2\mu_1 + G_1(\mathcal{I}A_1-1) + pG_2(\sum_m \Lambda_m A_2 - 1)}{r_b} \right) \end{aligned} \tag{78}$$

where

$$\begin{aligned} \Gamma_1 &= \left\{ w_1 - \frac{\lambda^2{}_{hb}-1}{2\theta} \right\} \mathcal{A}, \\ \Gamma_2 &= \left[\left\{ w_1 - \frac{\lambda^2{}_{hb}-1}{2\theta} \right\} \mathcal{B} + \frac{2\lambda_{hb}}{S'(\beta)} \left(1 + \frac{S(\beta)\lambda_{hb}}{S'(\beta)} \right) \mathcal{F} - \mathcal{K}_{\mathcal{L}}(n+\delta_2)\mathcal{F} \right], \\ \Gamma_3 &= \left[\left\{ w_1 - \frac{\lambda^2{}_{hb}-1}{2\theta} \right\} \mathcal{D} + \frac{2\lambda_{hb}}{S'(\beta)} \left(1 + \frac{S(\beta)\lambda_{hb}}{S'(\beta)} \right) \mathcal{E} + \mathcal{K}_{\mathcal{L}}(\kappa-\delta_2)\mathcal{E} \right], \\ \Gamma_4 &= \left[r_1 - \frac{\lambda^2{}_{k_1 b}-1}{2\pi} \right] K_1(0), \qquad \Gamma_5 \;=\; \left[\mathcal{K}_{\mathcal{C}}(\psi_{\mathcal{C}}+\delta_2) + g_c \right], \\ \Theta &= B(0) - \frac{\Gamma_1}{n+\gamma-r_b} + \frac{\Gamma_2}{n-r_b} - \frac{\Gamma_3}{\kappa+r_b} - \frac{\Gamma_4}{\psi_{k_1}-r_b} + \frac{\mathcal{C}_1(0)\Gamma_5}{\psi_{\mathcal{C}}-r_b} \\ & + \left(\frac{2\mu_1 + G_1(\mathcal{I}A_1-1) + pG_2(\sum_m \Lambda_m A_2 - 1)}{r_b} \right) \end{aligned}$$

6.7.2 Individual household's bond accumulation.

In Appendix A we show that the instantaneous budget constraint of household m in (1) yields the following differential equation in B_m.

$$\begin{aligned} \dot{B}_m(t) &= \mathcal{Z}_1 B_m(t) + \mathcal{Z}_2 \mathcal{A}_m e^{(n+\gamma)t} + \left[-\mathcal{Z}_2\mathcal{B}_m + \mathcal{Z}_3\mathcal{F}_m \right] e^{nt} + \mathcal{Z}_4 \mathcal{F} k_{2m}(0) e^{nt} + \left[-\mathcal{Z}_2\mathcal{D}_m + \mathcal{Z}_3\mathcal{E}_m \right] e^{-\kappa t} \\ & + \mathcal{Z}_6 \mathcal{E} k_{2m}(0) e^{-\kappa t} + \mathcal{Z}_7 K_{1m}(0) e^{\psi_{k_1} t} + \mathcal{Z}_8 \mathcal{C}_1(0) k_{2m}(0) e^{\psi_{\mathcal{C}} t} + \mathcal{Z}_9 \mathcal{C}_{1m}(0) e^{\psi_{\mathcal{C}} t} \\ & + \left[\mathcal{Z}_{10} - \frac{r_2 A_2 G_2 \sum_m \Lambda_m}{\alpha_2} \right] k_{2m}(0) + \mathcal{Z}_{11} + pA_2G_2\ \Lambda_m + T(t)\ Ind_m \end{aligned} \tag{79}$$

where

$$\begin{aligned} \mathcal{Z}_1 &= (1-\tau_b) r_b, \quad \mathcal{Z}_2 \;=\; \left[(1-\tau_w) w_1 - \frac{\lambda^2{}_{hb}-1}{2\theta} \right], \quad \mathcal{Z}_3 \;=\; w_2 - \frac{2\lambda_{hb}}{S'(\beta)} \left(1 + \frac{S(\beta)\lambda_{hb}}{S'(\beta)} \right), \qquad \mathcal{Z}_4 \;=\; (\\ \mathcal{Z}_{5am} &= -\mathcal{Z}_2\mathcal{B}_m + \mathcal{Z}_3\mathcal{F}_m, \qquad \mathcal{Z}_{5bm} \;=\; -\mathcal{Z}_2\mathcal{D}_m + \mathcal{Z}_3\mathcal{E}_m, \\ \mathcal{Z}_6 &= (\delta_2 - \kappa - r_2)\mathcal{K}_{\mathcal{L}}, \qquad \mathcal{Z}_7 \;=\; \left[(1-\tau_k) r_1 - \frac{\lambda^2{}_{k_1 b}-1}{2\pi} \right] \\ \mathcal{Z}_8 &= -(\psi_{\mathcal{C}} + \delta_2 - r_2)\mathcal{K}_{\mathcal{C}}, \qquad \mathcal{Z}_9 \;=\; -(1+\tau_c)(1+\frac{\eta_2}{\eta_1}) \\ \mathcal{Z}_{10} &= 2r_2\mathcal{K}_{\mathcal{M}}, \qquad \mathcal{Z}_{11} \;=\; (1+\tau_c)\mu_1 - \frac{\mu_2[(1-\tau_b)r_b + \delta_2]}{\alpha_2} \end{aligned}$$

6.7.3 The equilibrium trajectory of $B_1(t)$, $\mathcal{C}_{11}(0)$, and long-run growth of $B_1(t)$.

As noted in Section 2.3, the indicator function takes value $Ind_m = 0$ for $m = 1$. Hence, solving the differential equation (79) we get

$$\begin{aligned} B_1(t) \;=\; & \Bigg[B_1(0) - \Bigg\{\frac{\mathcal{Z}_2\mathcal{A}_1}{n+\gamma-\mathcal{Z}_1} + \frac{\mathcal{Z}_{5a1}}{n-\mathcal{Z}_1} + \frac{\mathcal{Z}_4\mathcal{F}k_{21}(0)}{n-\mathcal{Z}_1} - \frac{\mathcal{Z}_{5b1}}{\kappa+\mathcal{Z}_1} - \frac{\mathcal{Z}_6\mathcal{E}k_{21}(0)}{\kappa+\mathcal{Z}_1} \\ & + \frac{\mathcal{Z}_7K_{11}(0)}{\psi_{k_1}-\mathcal{Z}_1} + \frac{\mathcal{Z}_8k_{21}(0)\mathcal{C}_1(0)}{\psi_{\mathcal{C}}-\mathcal{Z}_1} + \frac{\mathcal{Z}_9\mathcal{C}_{11}(0)}{\psi_{\mathcal{C}}-\mathcal{Z}_1} - \left[\mathcal{Z}_{10} - \frac{r_2A_2G_2\sum_m\Lambda_m}{\alpha_2}\right]\frac{k_{21}(0)}{\mathcal{Z}_1} \\ & - \frac{(\mathcal{Z}_{11}+pA_2G_2\Lambda_1)}{\mathcal{Z}_1}\Bigg\}\Bigg]e^{\mathcal{Z}_1t} + \frac{\mathcal{Z}_2\mathcal{A}_1}{n+\gamma-\mathcal{Z}_1}e^{(n+\gamma)t} + \frac{\mathcal{Z}_{5a1}}{n-\mathcal{Z}_1}e^{nt} + \frac{\mathcal{Z}_4\mathcal{F}k_{21}(0)}{n-\mathcal{Z}_1}e^{nt} \\ & - \frac{\mathcal{Z}_{5b1}}{\kappa+\mathcal{Z}_1}e^{-\kappa t} - \frac{\mathcal{Z}_6\mathcal{E}k_{21}(0)}{\kappa+\mathcal{Z}_1}e^{-\kappa t} + \frac{\mathcal{Z}_7K_{11}(0)}{\psi_{k_1}-\mathcal{Z}_1}e^{\psi_{k_1}t} + \frac{\mathcal{Z}_8k_{21}(0)\mathcal{C}_1(0)}{\psi_{\mathcal{C}}-\mathcal{Z}_1}e^{\psi_{\mathcal{C}}t} + \frac{\mathcal{Z}_9\mathcal{C}_{11}(0)}{\psi_{\mathcal{C}}-\mathcal{Z}_1}e^{\psi_{\mathcal{C}}t} \\ & - \left[\mathcal{Z}_{10} - \frac{r_2A_2G_2\sum_m\Lambda_m}{\alpha_2}\right]\frac{k_{21}(0)}{\mathcal{Z}_1} - \frac{(\mathcal{Z}_{11}+pA_2G_2\Lambda_1)}{\mathcal{Z}_1} \end{aligned} \tag{80}$$

The transversality condition with respect to the bond holding for household 1 is

$$\lim_{t\to\infty} \lambda_{bm}(0).e^{-((1-\tau_b)r_b-\rho)t}e^{-\rho t}B_1(t) = 0 \quad\Longrightarrow\quad \lim_{t\to\infty} e^{-((1-\tau_b)r_b)t}B_1(t) = 0.$$

It follows from (80) that the transversality conditions for B_1 holds if the following conditions are true:

$$n+\gamma < (1-\tau_b)r_b \tag{81}$$
$$n < (1-\tau_b)r_b \tag{82}$$
$$\psi_{k_1} < (1-\tau_b)r_b \tag{83}$$
$$-\kappa < (1-\tau_b)r_b \tag{84}$$
$$\psi_{\mathcal{C}} < (1-\tau_b)r_b \tag{85}$$
$$(1-\tau_b)r_b > 0 \tag{86}$$

and

$$\begin{aligned} B_1(0) - \Bigg\{ & \frac{\mathcal{Z}_2\mathcal{A}_1}{n+\gamma-\mathcal{Z}_1} + \frac{\mathcal{Z}_{5a1}}{n-\mathcal{Z}_1} + \frac{\mathcal{Z}_4\mathcal{F}k_{21}(0)}{n-\mathcal{Z}_1} - \frac{\mathcal{Z}_{5b1}}{\kappa+\mathcal{Z}_1} - \frac{\mathcal{Z}_6\mathcal{E}k_{21}(0)}{\kappa+\mathcal{Z}_1} + \frac{\mathcal{Z}_7K_{11}(0)}{\psi_{k_1}-\mathcal{Z}_1} \\ & + \frac{\mathcal{Z}_8k_{21}(0)\mathcal{C}_1(0)}{\psi_{\mathcal{C}}-\mathcal{Z}_1} + \frac{\mathcal{Z}_9\mathcal{C}_{11}(0)}{\psi_{\mathcal{C}}-\mathcal{Z}_1} - \left[\mathcal{Z}_{10} - \frac{r_2A_2G_2\sum_m\Lambda_m}{\alpha_2}\right]\frac{k_{21}(0)}{\mathcal{Z}_1} \\ & - \frac{(\mathcal{Z}_{11}+pA_2G_2\Lambda_1)}{\mathcal{Z}_1}\Bigg\} = 0 \end{aligned} \tag{87}$$

From equation (87), the initial consumption level of the formal sector household is obtained as:

$$\begin{aligned}\mathcal{C}_{11}(0) &= \frac{\psi_{\mathcal{C}} - \mathcal{Z}_1}{\mathcal{Z}_9}\Bigg[B_1(0) - \Bigg\{\frac{\mathcal{Z}_2\mathcal{A}_1}{n+\gamma-\mathcal{Z}_1} + \left(\frac{-\mathcal{Z}_2\mathcal{B}_1 + \mathcal{Z}_3\mathcal{F}_1}{n-\mathcal{Z}_1}\right) \\ &\quad - \left(\frac{-\mathcal{Z}_2\mathcal{D}_1 + \mathcal{Z}_3\mathcal{E}_1}{\kappa+\mathcal{Z}_1}\right) + \frac{\mathcal{Z}_7 K_{11}(0)}{\psi_{k_1} - \mathcal{Z}_1} \\ &\quad + k_{21}(0)\left(\frac{\mathcal{Z}_4\mathcal{F}}{n-\mathcal{Z}_1} - \frac{\mathcal{Z}_6\mathcal{E}}{\kappa+\mathcal{Z}_1} + \frac{\mathcal{Z}_8\mathcal{C}_1(0)}{\psi_{\mathcal{C}}-\mathcal{Z}_1} - \left[\mathcal{Z}_{10} - \frac{r_2 A_2 G_2 \sum_m \Lambda_m}{\alpha_2}\right]\frac{1}{\mathcal{Z}_1}\right) \\ &\quad - \frac{(\mathcal{Z}_{11} + pA_2G_2\Lambda_1)}{\mathcal{Z}_1}\Bigg\}\Bigg] \end{aligned} \tag{88}$$

Using conditions (81) to (87) and (80) the equilibrium trajectory of bond accumulation by the formal sector can be derived and from that it follows that the long-run growth rate of $B_1(t)$ is

$$\lim_{t\longrightarrow\infty}\frac{\dot{B}_1(t)}{B_1(t)} = \max\ \{\ n+\gamma,\ n,\ \psi_{\mathcal{C}},\ \psi_{k_1}\} \tag{89}$$

that is, it is the maximum of the long-run rates of growth of the human capital, formal-sector physical capital, population growth rate of the unskilled labour force, and consumption.

6.7.4 $B_2(t)$ and the transversality condition condition.

Recalling the definition of $\mathcal{C}_1$ in (70), and knowing the initial values $\mathcal{C}_1(0)$ and $\mathcal{C}_{11}(0)$ from (77) and (88), we obtain the initial value $\mathcal{C}_{12}(0)$ as

$$\mathcal{C}_{12}(0) \quad = \quad \mathcal{C}_1(0) - \mathcal{C}_{11}(0) \tag{90}$$

Substituting for the equilibrium trajectories of $H(t)$ and $C_1(t)$ into the government's budget condition (21), the trajectory of transfer to the informal sector is obtained as

$$\begin{aligned} T(t) &= \tau_b r_b B(t) + \tau_k r_1 K_1(0)e^{\psi_{k_1}t} + \tau_w w_1\left(\mathcal{A}e^{(n+\gamma)t} - \mathcal{B}e^{nt} + \mathcal{D}\kappa e^{-\kappa t}\right) + \tau_c\left(\mathcal{C}_1(0)e^{\psi_{\mathcal{C}}t} + 2\mu_1\right) \\ &\quad - G_1(1-\mathcal{I}A_1) - g_c\mathcal{C}_1(0)e^{\psi_{\mathcal{C}}t} - pG_2 \end{aligned}$$

Then the individual household bond accumulation equation (79) for household 2 becomes

$$\begin{aligned}\dot{B}_2(t) &= (1-\tau_b)r_b B_2(t) + \mathcal{Z}_2\mathcal{A}_2 e^{(n+\gamma)t} + \left[-\mathcal{Z}_2\mathcal{B}_2 + \mathcal{Z}_3\mathcal{F}_2\right]e^{nt} + \mathcal{Z}_4\mathcal{F}k_{22}(0)e^{nt} \\ &+ \left[-\mathcal{Z}_2\mathcal{D}_2 + \mathcal{Z}_3\mathcal{E}_2\right]e^{-\kappa t} + \mathcal{Z}_6\mathcal{E}k_{22}(0)e^{-\kappa t} \\ &+ \mathcal{Z}_7 K_{12}(0)e^{\psi_{k_1}t} + \mathcal{Z}_8\mathcal{C}_1(0)k_{22}(0)e^{\psi_{\mathcal{C}}t} + \mathcal{Z}_9\mathcal{C}_{12}(0)e^{\psi_{\mathcal{C}}t} + \left[\mathcal{Z}_{10} - \frac{r_2 A_2 G_2\sum_m \Lambda_m}{\alpha_2}\right]\frac{k_{22}(0)}{\mathcal{Z}_1} \\ &+ \mathcal{Z}_{11} - G_1(1-\mathcal{I}A_1) - pG_2(1-\Lambda_2 A_2) + 2\tau_c\mu_1 + \tau_b r_b B(t) \\ &+ \tau_k r_1 K_1(0)e^{\psi_{k1}t} + \tau_w w_1\left(\mathcal{A}e^{(n+\gamma)t} - \mathcal{B}e^{nt} + \mathcal{D}\kappa e^{-\kappa t}\right) + (\tau_c - g_c)\mathcal{C}_1(0)e^{\psi_{\mathcal{C}}t} \qquad (91)\end{aligned}$$

where $B(t)$ is given from equation (78). In the Appendix C we show that by solving the above differential equation, the transversality condition can be derived for B_2 as in the case of B_1. This condition would be satisfied if (81) to (86) are true and the following hold:

$$\begin{aligned}B_2(0) - \Bigg\{&\frac{\mathcal{Z}_2\mathcal{A}_2}{n+\gamma-\mathcal{Z}_1} + \frac{\mathcal{Z}_{5a2}}{n-\mathcal{Z}_1} + \frac{\mathcal{Z}_4\mathcal{F}k_{22}(0)}{n-\mathcal{Z}_1} - \frac{\mathcal{Z}_{5b2}}{\kappa+\mathcal{Z}_1} - \frac{\mathcal{Z}_6\mathcal{E}k_{22}(0)}{\kappa+\mathcal{Z}_1} + \frac{\mathcal{Z}_7 K_{12}(0)}{\psi_{k_1}-\mathcal{Z}_1} \\ &+ \frac{\mathcal{Z}_8 k_{22}(0)\mathcal{C}_1(0)}{\psi_{\mathcal{C}}-\mathcal{Z}_1} + \frac{\mathcal{Z}_9\mathcal{C}_{12}(0)}{\psi_{\mathcal{C}}-\mathcal{Z}_1} - \left[\mathcal{Z}_{10} - \frac{r_2 A_2 G_2\sum_m\Lambda_m}{\alpha_2}\right]\frac{k_{22}(0)}{\mathcal{Z}_1} \\ &+ \frac{\tau_k r_1 K_1(0)}{\psi_{k_1}-r_b} + \frac{\tau_w w_1\mathcal{A}}{n+\gamma-r_b} - \frac{\tau_w w_1\mathcal{B}}{n-r_b} - \frac{\tau_w w_1\kappa\mathcal{D}}{\kappa+r_b} + \frac{(\tau_c-g_c)\mathcal{C}_1(0)}{\psi_{\mathcal{C}}-r_b} \\ &- \frac{(\mathcal{Z}_{11} - G_1(1-\mathcal{I}A_1) - pG_2(1-\Lambda_2 A_2) + 2\tau_c\mu_1)}{r_b} \\ &+ \frac{\tau_b r_b\Gamma_1}{(n+\gamma-r_b)(n+\gamma-\mathcal{Z}_1)} - \frac{\tau_b r_b\Gamma_2}{(n-r_b)(n-\mathcal{Z}_1)} \\ &- \frac{\tau_b r_b\Gamma_3}{(\kappa+r_b)(\kappa+\mathcal{Z}_1)} + \frac{\tau_b r_b\Gamma_4}{(\psi_{k_1}-r_b)(\psi_{k_1}-\mathcal{Z}_1)} \\ &- \frac{\tau_b r_b\mathcal{C}_1(0)\Gamma_5}{(\psi_{\mathcal{C}}-r_b)(\psi_{\mathcal{C}}-\mathcal{Z}_1)} + \tau_b\left(\frac{2\mu_1 + G_1(\mathcal{I}A_1-1) + pG_2(\Lambda_2 A_2 - 1)}{\mathcal{Z}_1}\right)\Bigg\} = 0 \qquad (92)\end{aligned}$$

and

$$\begin{aligned}\Theta &= B(0) - \frac{\Gamma_1}{n+\gamma-r_b} + \frac{\Gamma_2}{n-r_b} - \frac{\Gamma_3}{\kappa+r_b} - \frac{\Gamma_4}{\psi_{k_1}-r_b} + \frac{\mathcal{C}_1(0)\Gamma_5}{\psi_{\mathcal{C}}-r_b} \\ &+ \left(\frac{2\mu_1 + G_1(\mathcal{I}A_1-1) + pG_2(\sum_m\Lambda_m A_2 - 1)}{r_b}\right) = 0 \qquad (93)\end{aligned}$$

Under these conditions, the long-run growth rate of B_2 is the same as the long-run growth rate of B_1, *i.e.*, it is given by

$$\lim_{t\longrightarrow\infty}\frac{\dot{B}_2(t)}{B_2(t)} = \max\left\{\, n+\gamma,\ n,\ \psi_{\mathcal{C}},\ \psi_{k_1}\right\} \qquad (94)$$

Note too that, combined with the levels of the initial endowments and the knowledge of

$\mathcal{C}_1(0)$ from (77), $\mathcal{C}_{11}(0)$ from (88), and hence $\mathcal{C}_{12}(0)$ from (90), the two equations (92) and (93) can be employed to solve for the equilibrium values of the public infrastructure expenditures G_1 and G_2 in the formal and the informal sectors. Thus, G_1 and G_2 are endogenously determined at the macroeconomic tax equilibrium of our model.

6.7.5 The transversality condition for K_{2m} and its long-run growth rate.

The transversality condition for physical capital in the informal sector is

$$\lim_{t\to\infty} \lambda_{k_2m}(t)K_{2m}(t)e^{-\rho t} = 0.$$

Given (7) this implies that

$$\lim_{t\to\infty} \lambda_{bm}(t)K_{2m}(t)e^{-\rho t} = 0 \quad \Longrightarrow \quad \lambda_{bm}(0)\lim_{t\to\infty} K_{2m}(t)e^{-(1-\tau_b)r_b t} = 0 \tag{95}$$

where, employing (71) and (73), it follows that

$$K_{2m}(t) \;\; = \;\; k_{2m}(0)\Big(\mathcal{K}_{\mathcal{C}}\mathcal{C}_1(t) - \mathcal{K}_{\mathcal{L}}L(t) + 2\mathcal{K}_{\mathcal{M}} - \frac{A_2G_2\sum_m \Lambda_m}{\alpha_2}\Big)$$

It can be seen that the transversality condition holds for K_{2m} if the transversality condition holds for B_m. Furthermore, (73) implies that the long-run growth rate of K_{2m} is given by

$$\lim_{t\longrightarrow\infty} \frac{\dot{K}_{2m}(t)}{K_{2m}(t)} = \max\;\{\;n,\;\psi_{\mathcal{C}}\} \tag{96}$$

Thus, the growth rate of the informal sector physical capital is determined by the maximum of the consumption and the unskilled labour population growth rates.

6.8 Longrun equilibrium rates of growths of outputs.

From the technological specification of the formal sector producers in Section 4 it follows that in equilibrium, we have $Y_1(t) = \alpha_1K_1(t)+\alpha_HH(t)+A_1G_1$. Since G_1 is an endogenously determined constant, the long-run rate of growth of Y_1 is

$$\lim_{t\to\infty} \frac{\dot{Y}_1(t)}{Y_1(t)} \;\; = \;\; \max\{\psi_{k_1},\;n,\;n+\gamma\} \tag{97}$$

Similarly, the technological assumption for the technology used in the informal sector implies that it in equilibrium, we have $Y_2(t) = \alpha_2 K_2(t) + \alpha_L L(t) + A_2 G_2$. Given that G_2 is a fixed constant that is endogenously determined, the long-run rate of growth of Y_2 is the same as the long-run rate of growth of K_2:

$$\lim_{t\to\infty} \frac{\dot{Y}_2(t)}{Y_2(t)} = \max\ \{n,\ \psi_{\mathcal{C}}\} \tag{98}$$

6.9 Unbalanced growth and the equilibrium values of $\mathcal{C}_1(0)$, $\mathcal{C}_{11}(0)$, $\mathcal{C}_{12}(0)$, G_1, and G_2.

From the above analysis it follows that the equilibrium values of $\mathcal{C}_1(0)$, $\mathcal{C}_{11}(0)$, $\mathcal{C}_{12}(0)$, G_1, and G_2 are determined endogenously by the system of five equations (77), (88), (90), (93), and (92). Solution to this system of equations fixes the macro-economic decentralised equilibrium trajectories of all state variables and consumption and migration including the long-run growth rates of these variables.

The rates of growth both at a given point in time and in the long-run of various macroeconomic variables at a macroeconomic tax equilibrium were derived in the previous sections. We saw that there are no transitional dynamics for economic variables such as the relative shadow prices of all types of capital and the net labour employed in the informal sector (which turns out to be like another state variable in our dynamic model), share of migration in the net labour force employed in the informal sector, and the level of skill acquisition by the migrating labour force. Equilibrium values of these are constant over time.

The rates of growth of the formal sector physical capital, K_1, and $\mathcal{C}_{1m}$ for $m = 1, 2$ are constant over time and determined by different factors. While the rate of growth of $\mathcal{C}_{1m}$, denoted by $\psi_{\mathcal{C}}$, is determined by the difference in the net-of-tax return on the bond, $(1 - \tau_b)r_b$, and the rate of time preference, ρ, the rate of growth of the formal-sector physical capital, denoted by ψ_{k_1}, is determined by its relative shadow price, λ_{hb}.

In Section 6.1 we saw that while consumption of the formal and informal sector goods, C_1 and C_2, exhibited transitional dynamics in a macroeconomic tax equilibrium, their long-run rates of growth were the same and equal to $\psi_{\mathcal{C}}$.

Transitional dynamics were also exhibited by the supply of net unskilled labour in the informal sector, L, the stocks of human capital, H, and the informal-sector physical capital, K_2, and the holdings of bond B by the two households.

In the long run, however, the rate of growth of unskilled labour is given by the exogenous population growth rate n. This is also the long-run rate of growth of migration, given that the share of migration in the net labour force employed in the informal sector is a constant.

The long-run rate of growth of the stock of human capital is determined by the greater of the long-run rate of migration, n, and investment in human capital as a proportion of the total stock of human capital, which in turn is a fixed constant determined by the relative shadow price of human capital, denoted by λ_{hb}.

Under the maintained assumptions of our model, the long-run rate of growth of physical capital engaged in the informal sector, denoted by ψ_{k_2} is the bigger of the long-run rate of growth of net labour force employed in this sector, given by n, and the long-run rate of growth of consumption $\psi_{\mathfrak{C}}$.

The long-run rate of bond accumulation is obtained as the greater of the long-run rates of growth of human capital and physical capital employed in the formal and informal sectors.

While the long-run rate of growth of the formal sector output is determined by the maximum of the long-run rates of growth of physical or human capital employed in this sector, the long-run rate of growth of output produced in the informal sector is determined by the maximum of the long-run rates of growth of unskilled labour and consumption.

7 Social welfare.

Given the vector of parameters in our analysis

$$\widehat{\mathcal{P}} = \langle \rho, \epsilon, \varphi, \delta_{k_1}, \delta_{k_2}, \theta, \pi, A_1, A_2, \alpha_1, \alpha_2, \alpha_H, \alpha_L, \eta_2, \eta_1, \mu_1, \mu_2, r_b, n, g_c \rangle$$

and the exogenous distribution of initial endowments vectors

$$\langle K_{1m}(0), K_{2m}(0), l_m(0), L_m(0), H_m(0), B_m(0) \rangle \ \forall \ m = 1, 2$$

a macroeconomic tax equilibrium can be computed for every vector of tax rates $\tau = \langle \tau_w, \tau_b, \tau_k, \tau_c \rangle$ following the methodology provided in the previous section. We compute below the welfare of individual intertemporal utility maximising households and the social welfare based on an individualistic social welfare function at the macroeconomic tax equilibrium associated with every vector of tax rates. The social welfare maximisation problem then boils down to finding that vector of tax rates and the associated macroeconomic tax equilibrium that leads to highest social welfare.

7.1 Household's intertemporal welfare.

The household's intertemporal welfare for the Stone Geary preference structure used in this analysis (see Section 6.1) at a decentralised equilibrium of our model is obtained as follows: [21]

$$
\begin{aligned}
u_m &= \int_0^\infty e^{-\rho t} U\left(C_{1m}(t), C_{2m}(t), G_c(t)\right) e^{-\rho t} dt \\
&= \left(\frac{1}{1-\epsilon}\right)\left(\frac{\eta_2(1+\tau_c)}{\eta_1 p}\right)^{\eta_2(1-\epsilon)} \left(g_c \mathcal{C}_1(0)\right)^{1-\epsilon} \left(\frac{1}{\rho - 2\psi_{\mathcal{C}}(1-\epsilon)}\right) \mathcal{C}_{1m}(0)^{1-\epsilon} \\
&=: \mathcal{U}\left(\mathcal{C}_{1m}(0), \mathcal{C}_1(0), \tau, \widehat{\mathcal{P}}\right) \qquad (99)
\end{aligned}
$$

where, p is given from (22). Recalling (29), which derives the growth rate of $\mathcal{C}_1$, $\psi_{\mathcal{C}}$, it is clear that (99) is well-defined only if

$$
2\psi_{\mathcal{C}}(1-\epsilon) - \rho < 0 \quad \Longrightarrow \quad \left[\frac{(1-\tau_b)r_b - \rho}{\epsilon}\right] 2(1-\epsilon) - \rho < 0 \qquad (100)
$$

7.2 Isoelastic social welfare.

We consider an individualistic social welfare function, where social welfare is an isoelastic function of the welfare of the two households:

$$
W(u_1, u_2) = \frac{1}{1-\varphi} \sum_{m=1}^{2} \left[(u_m)^{1-\varphi} - 1\right], \; 0 \le \varphi < \infty, \; \varphi \neq 1.
$$

Here, φ is the parameter of inequality aversion. Some special cases of the social welfare function W include the Benthamite/utilitarian case where $\varphi = 0$ (society is least inequality averse) and the Rawlsian case when $\varphi = \infty$ (the case where there is maximum inequality aversion). Employing (99), we obtain social welfare at a macroeconomic tax equilibrium as the function

$$
\begin{aligned}
W(u_1, u_2) &= W(\mathcal{U}\left(\mathcal{C}_{11}(0), \mathcal{C}_1(0), \tau, \widehat{\mathcal{P}}\right), \mathcal{U}\left(\mathcal{C}_{12}(0), \mathcal{C}_1(0), \tau, \widehat{\mathcal{P}}\right)) \\
&= \frac{1}{1-\varphi}\left[\left(\frac{1}{1-\epsilon}\right)\left(\frac{\eta_2(1+\tau_c)}{\eta_1 p}\right)^{\eta_2(1-\epsilon)} \left(g_c\, \mathcal{C}_1(0)\right)^{1-\epsilon} \left(\frac{1}{\rho - 2\psi_{\mathcal{C}}(1-\epsilon)}\right)\right]^{1-\varphi} \\
&\qquad \times \sum_{m=1}^{2} \mathcal{C}_{1m}(0)^{(1-\epsilon)(1-\varphi)} \\
&=: \bar{\mathcal{W}}\left(\mathcal{C}_{11}(0), \mathcal{C}_{12}(0), \mathcal{C}_1(0), \tau, \widehat{\mathcal{P}}\right), \qquad (101)
\end{aligned}
$$

[21] See the Appendix B for detailed derivation.

where $\mathcal{C}_{11}(0)$ is obtained from (88), $\mathcal{C}_1(0)$ is obtained from (77), and $\mathcal{C}_{12}(0) = \mathcal{C}_1(0) - \mathcal{C}_{11}(0)$.

7.3 Social welfare maximisation.

The social welfare maximisation problem in this analysis takes the form of choosing the various tax rates $\tau_b,\ \tau_k,\ \tau_w$, and τ_c lying in the interval $[0,1]$ and associated non-negative vector of other fiscal policy and relevant economic variables $\langle G_1, G_2, \mathcal{C}_{11}(0), \mathcal{C}_{12}(0), \mathcal{C}_1(0), \lambda_{hb}, \lambda_{k_1 b}, \kappa, \beta, \gamma, \psi_{k_1}, \psi_{\mathcal{C}}\rangle$ to maximise (101) subject to the following constraints:[22]

i) $\langle G_1, G_2, \mathcal{C}_{11}(0), \mathcal{C}_{12}(0), \mathcal{C}_1(0)\rangle$ solve the system of equilibrium equations (77) to (92)

ii) the common long-run equilibrium rates of growth of consumption of both the formal and informal sector goods is

$$\psi_{\mathcal{C}} = \frac{(1-\tau_b)r_b - \rho}{\epsilon}$$

iii) Condition (100) for intertemporal welfare of each household to be well-defined holds, *i.e.*,

$$2\psi_{\mathcal{C}}(1-\epsilon) - \rho < 0$$

iv) λ_{hb} and $\lambda_{k_1 b}$ are real-valued negative roots of (41) and (33), respectively, *i.e.*,

$$\begin{aligned} \lambda_{hb} &= \Big(1 + (1-\tau_b)r_b\theta\Big) - \sqrt{(1-\tau_b)^2 r_b^2\theta^2 + 2\theta(1-\tau_b)r_b - 2\theta(1-\tau_w)w_1} \\ \lambda_{k_1 b} &= 1 + \pi\delta_{k_1} + \pi(1-\tau_b)r_b - \sqrt{\left(1 + \pi\delta_{k_1} + \pi(1-\tau_b)r_b\right)^2 - \left(1 + 2\pi r_1(1-\tau_k)\right)} \end{aligned}$$

with

$$(1-\tau_b)^2 r_b^2\theta^2 + 2\theta(1-\tau_b)r_b - 2\theta(1-\tau_w)\alpha_H \geq 0$$
$$\left(1 + \pi\delta_{k_1} + \pi(1-\tau_b)r_b\right)^2 - \left(1 + 2\pi\alpha_1(1-\tau_k)\right) \geq 0.$$

v) β is obtained by solving

$$\frac{2S(\beta_m)}{S'(\beta_m)^2}\lambda_{hbm}^2 - \left[\frac{\beta_m S'(\beta_m) - 2S(\beta_m)}{S'(\beta_m)}\right](1-\tau_b)r_b\lambda_{hbm} + \Big((1-\tau_b)r_b + w_2\Big) = 0$$

[22]The transfer to the informal sector $T(t) \geq 0$ – also a fiscal instrument – adjusts to ensure that the government's budget balance condition holds at any time point t.

vi) κ is obtained as

$$\kappa = \frac{2\lambda_{hb}}{S'(\beta)}$$

vii) The transversality condition for L_m is true for $m = 1, 2$:

$$n - (1 - \tau_b) r_b \leq 0.$$

viii) γ is obtained as

$$\gamma = \frac{\lambda_{hb} - n\theta - 1}{\theta}$$

ix) The long-run rate of growth of formal sector capital ψ_{k_1} is given by

$$\begin{aligned}
\psi_{k_1} &= \frac{\pi(1-\tau_b)r_b \pm \sqrt{(1+\pi\delta_{k_1}+\pi(1-\tau_b)r_b)^2 - (1+2\pi r_1(1-\tau_k))}}{\pi} \\
&= \frac{\pi(1-\tau_b)r_b \pm \sqrt{(\pi\delta_{k_1}+\pi(1-\tau_b)r_b)^2 + 2\left(\pi\delta_{k_1}+\pi(1-\tau_b)r_b\right) - 2\pi r_1(1-\tau_k)}}{\pi}
\end{aligned}$$

Though in constraint (iv) above we have chosen the negative roots of (41) and (33), we still need to ensure that λ_{hb} and $\lambda_{k_1 b}$ are non-negative. We find that these are automatically satisfied as:

$$\begin{aligned}
\lambda_{hb} &= \Big(1 + (1-\tau_b)r_b\theta\Big) - \sqrt{(1-\tau_b)^2 r_b^2\theta^2 + 2\theta(1-\tau_b)r_b - 2\theta(1-\tau_w)w_1} \;\geq\; 0 \\
&\implies 1 \geq -2\theta(1-\tau_w)w_1 \text{ which is always true.}
\end{aligned}$$

and

$$\begin{aligned}
\lambda_{k_1 b} &= 1 + \pi\delta_{k_1} + \pi(1-\tau_b)r_b - \sqrt{(1+\pi\delta_{k_1}+\pi(1-\tau_b)r_b)^2 - (1+2\pi r_1(1-\tau_k))} \;\geq\; 0 \\
\implies & \quad 1 + \pi\delta_{k_1} + \pi(1-\tau_b)r_b + \pi\alpha_1(1-\tau_k) \geq 0 \text{ which is always true.}
\end{aligned}$$

We show below that constraints (i) to (ix) above also ensure that all the transversality conditions for all the state variables hold. These transversality conditions are summarised by (81) to (87):[23]

[23]These are also the complete set of transversality conditions for B_m. These also ensure that transversality conditions hold for K_{2m}.

- The transversality condition for H required that

$$\begin{aligned}\gamma + n - (1-\tau_b)r_b &= \frac{\lambda_{hb} - 1 - (1-\tau_b)r_b\theta}{\theta} \\ &= \pm\frac{\sqrt{\theta\left(2(1-\tau_b)r_b + (1-\tau_b)^2 r_b^2\theta - 2(1-\tau_w)w_1\right)}}{\theta} < 0\end{aligned}$$

This is true if and only if the negative root of λ_{hb} is chosen and constraint (iv) is true, that is,

$$\theta\left(2(1-\tau_b)r_b + (1-\tau_b)^2 r_b^2\theta - 2(1-\tau_w)w_1\right) \geq 0$$

- The transversality condition for L required that

$$n - (1-\tau_b)r_b < 0$$

which is constraint (vii).

- The transversality condition for K_1 requires that

$$\psi_{k_1} - (1-\tau_b)r_b < 0 \iff \frac{\lambda_{k_1 b} - 1 - \delta_{k_1}\pi - (1-\tau_b)r_b\pi}{\pi} < 0$$

This is true if and only if the negative root of $\lambda_{k_1 b}$ is chosen and constraint (iv) above is true, that is,

$$\left(1 + \pi\delta_{k_1} + \pi(1-\tau_b)r_b\right)^2 - \left(1 + 2\pi\alpha_1(1-\tau_k)\right) \geq 0$$

- Additionally, the transversality conditions for B_m require

$$\begin{aligned}(1-\tau_b)r_b &> 0 \\ -\kappa &< (1-\tau_b)r_b \\ \psi_{\mathcal{C}} &< (1-\tau_b)r_b\end{aligned}$$

The first is ensured by assuming $r_b \geq 0$ and $\tau_b \in [0,1]$. The second holds automatically given the first as κ is expected to be non-negative. The third condition is (85) and is true when constraint (iii) above is true: This is because (85) implies that

$$\frac{(1-\tau_b)r_b(1-\epsilon)}{\epsilon} < \rho, \qquad (102)$$

while constraint (iii) implies that

$$\frac{(1-\tau_b)r_b(1-\epsilon)}{\epsilon} < \rho\left(\frac{2-\epsilon}{2}\right) \tag{103}$$

It is clear that (103) implies (102) if and only if $\frac{2-\epsilon}{2} < 1$. This is true as long as $\epsilon > 0$, which is a maintained assumption with respect to our preferences defined in Section 2.2.

8 Conclusions.

In this work, we propose a model of a dual economy that has features of a contemporary developing economy that has, to a large extent, overcome the historical shackles of social and cultural restrictions on occupational mobility and where education has become more accessible to all sections of the society. A modern-day labour-rich developing economy is characterised by migration of labour from the rural informal (agrarian) sector, with the proviso that the migrating labour force can contribute to the pool of skilled labour resource in the urban (manufacturing and services) formal sector by investing in education and skill formation. The more the migratory labour force will spend on skill formation, the more it will contribute to the human capital in the formal sector. This, coupled with qualitative differences in the physical capital employed in the formal and informal sectors, the Engel's-law based demand patterns for goods produced in these sectors, and the fact that the consumption of the informal/agricultural good above a subsistence level is essential for both types of households, this implies a dual structure for the economy that will persist as long it continues to be characterised by a predominance of unskilled labour resource due to high growth rates of population in the working age group, a phenomenon referred to as a demographic dividend that is a feature of many contemporary developing economies.

Production in both sectors can be enhanced by government infrastructural expenditures and it is possible for both sectors to grow endogenously; albeit, the dual character of the developing economy implies unbalanced sectoral growths. While the long-run growth rate of the informal sector output in our model is determined by the bigger of the growth rate of the woking population and the long-run rate of growth of consumption; [24] the long-run rate of growth of output in the formal sector is the bigger of the long-run growth rates of consumption, population, and human and physical capital formation in the formal sector.

[24] While in the short-run the two consumption goods – the goods produced in the informal and formal sectors – grow at different rates, their long-run growth rates are the same.

Inequalities that arise between the formal and informal sector households on account of differences in their initial endowments and incomes due to the unbalanced sectoral growth can be mitigated by sound fiscal policies. Assuming that the representative household in the formal sector is more richly endowed than its informal sector counterpart, government's redistributive and expenditure policies include taxation of both physical and human capital in the formal sector, taxation of profit/economic rent generated by government infrastructural activities in the formal sector, and taxation of consumption by both types of households with the total tax revenue so generated financing a transfer to the informal sector household, public infrastructure expenditures in the two sectors, and a public good consumed by both types of households. The household is assumed to be self employed in the informal sector, so that all profit/economic rent from use of public infrastructure in the informal sector accrue as income to the household.

In this study, we first define a macroeconomic tax equilibrium for every configuration of tax rates (capital and consumption tax rates). In the particular model of growth that we have set up, the equilibrium levels of migration and skills chosen by the migrating labour force are determined endogenously as are also the equilibrium levels of government's infrastructural expenditures in the formal and informal sectors. The inter-temporal welfare of both formal and informal sector households at a tax equilibrium are derived. Adopting an iso-elastic social welfare function, the work then poses the social welfare maximisation problem for determining the optimal tax rates, levels of government expenditures, migration, and skill formation by the migratory labour force.

In the sequel to this paper (Das and Murty (2022)), the theory developed here is subjected to some detailed and rigorous numerical simulations to get a flavour of the nature of fiscal policy prescriptions generated by this model of a contemporary dual economy. These numerical simulations address questions such as: Which type of capital – human or physical– is to be subjected to taxation? What is the nature of substitutability between physical migration and skill formation by migratory labour force in contributing to effective units of skilled labour/human capital to the formal sector? What are the impacts of changes in labour and physical capital productivity differentials between the formal and informal sectors on optimal fiscal policies? How successful is the transfer to the informal sector household in achieving redistribution and greater equity? Is consumption taxation desirable? How are the trade-offs between growth and redistribution resolved by optimal fiscal policies?

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APPENDIX

A Deriving the differential equation for household bond accumulation $\dot{B}_m$

The instantaneous budget constraint of household m in (1) yields

$$\begin{aligned}\dot{B}_m &= (1-\tau_w)w_1 H_m + w_2 L_m + (1-\tau_k)r_1 K_{1m} + r_2 K_{2m} + (1-\tau_b)r_b B_m \\ &\quad -(1+\tau_c)C_{1m} - pC_{2m} - X_m\left(1+\frac{S(\beta)X_m}{2L_m}\right) - I_{k_1m}\left(1+\frac{\pi I_{k_1m}}{2K_{1m}}\right) \\ &\quad -I_{k_{2m}} - I_{hm}\left(1+\frac{\theta I_{hm}}{2H_m}\right) + T\ Ind_m + pA_2G_2\ \Lambda_m\end{aligned}$$

Recalling that $C_{1m} = \mathcal{C}_{1m} - \mu_1$ and $C_{2m} = \mathcal{C}_{2m} + \mu_2$ and noting the relation between $\mathcal{C}_{1m}$ and $\mathcal{C}_{2m}$ in (27), this yields

$$\begin{aligned}\dot{B}_m &= (1-\tau_w)w_1 H_m + w_2 L_m + (1-\tau_k)r_1 K_{1m} + r_2 K_{2m} + (1-\tau_b)r_b B_m \\ &\quad -(1+\tau_c)(\mathcal{C}_{1m} - \mu_1) - p(\mathcal{C}_{2m} + \mu_2) - X_m\left(1+\frac{S(\beta)X_m}{2L_m}\right) \\ &\quad -I_{k_1m}\left(1+\frac{\pi I_{k_1m}}{2K_{1m}}\right) - I_{k_{2m}} - I_{hm}\left(1+\frac{\theta I_{hm}}{2H_m}\right) + T\ Ind_m + pA_2G_2\ \Lambda_m\end{aligned}$$

which implies

$$\begin{aligned}\dot{B}_m &= (1-\tau_w)w_1 H_m + w_2 L_m + (1-\tau_k)r_1 K_{1m} + r_2 K_{2m} + (1-\tau_b)r_b B_m \\ &\quad -(1+\tau_c)\mathcal{C}_{1m} + (1+\tau_c)\mu_1 - p\chi_1\mathcal{C}_{1m} - p\mu_2 - X_m\left(1+\frac{S(\beta)X_m}{2L_m}\right) \\ &\quad -I_{k_1m}\left(1+\frac{\pi I_{k_1m}}{2K_{1m}}\right) - I_{k_{2m}} - I_{hm}\left(1+\frac{\theta I_{hm}}{2H_m}\right) + T\ Ind_m + pA_2G_2\ \Lambda_m\end{aligned}$$

Collecting terms and re-arranging we obtain

$$\begin{aligned}\dot{B}_m &= (1-\tau_b)r_b B_m + H_m\left[(1-\tau_w)w_1 - \frac{I_{hm}}{H_m}\left(1+\frac{\theta I_{hm}}{2H_m}\right)\right] + K_{1m}\left[(1-\tau_k)r_1\right. \\ &\quad \left.-\frac{I_{k_1m}}{K_{1m}}\left(1+\frac{\pi I_{k_1m}}{2K_{1m}}\right)\right] + L_m\left[w_2 - \frac{X_m}{L_m}\left(1+\frac{S(\beta)X_m}{2L_m}\right)\right] + r_2K_{2m} - I_{k_{2m}} \\ &\quad +(1+\tau_c)\mu_1 - (1+\tau_c)(1+\frac{\eta_2}{\eta_1})\mathcal{C}_{1m} - \frac{\mu_2[(1-\tau_b)r_b + \delta_2]}{\alpha_2} + T\ Ind_m + pA_2G_2\ \Lambda_m\end{aligned}$$

Using (6) and (8) we obtain

$$\dot{B}_m = (1-\tau_b)r_b B_m + H_m\left[(1-\tau_w)w_1 - \frac{\lambda^2{}_{hbm}-1}{2\theta}\right] + K_{1m}\left[(1-\tau_k)r_1 - \frac{\lambda^2{}_{k_1bm}-1}{2\pi}\right]$$
$$+L_m\left[w_2 - \frac{2\lambda_{hb}}{S'(\beta)}\left(1+\frac{S(\beta)\lambda_{hb}}{S'(\beta)}\right)\right] + r_2K_{2m} - I_{k_{2m}}$$
$$+(1+\tau_c)\mu_1 - (1+\tau_c)(1+\frac{\eta_2}{\eta_1})\mathcal{C}_{1m} - \frac{\mu_2[(1-\tau_b)r_b+\delta_2]}{\alpha_2} + T\ Ind_m + pA_2G_2\ \Lambda_m$$

Employing (73) and (75) we obtain

$$\dot{B}_m(t) = (1-\tau_b)r_bB_m(t) + \left[(1-\tau_w)w_1 - \frac{\lambda^2{}_{hb}-1}{2\theta}\right]\left[e^{(n+\gamma)t}\mathcal{A}_m - e^{nt}\mathcal{B}_m - e^{-\kappa t}\mathcal{D}_m\right]$$
$$+\left[(1-\tau_k)r_1 - \frac{\lambda^2{}_{k_1b}-1}{2\pi}\right]K_{1m}(0)e^{\psi_{k_1}t} + \left[w_2 - \frac{2\lambda_{hb}}{S'(\beta)}\left(1+\frac{S(\beta)\lambda_{hb}}{S'(\beta)}\right)\right]\ \left[e^{-\kappa t}\mathcal{E}_m + e^{nt}\mathcal{F}_m\right]$$
$$+r_2k_{2m}(0)\left(\mathcal{K}_{\mathcal{C}}\mathcal{C}_1(t) - \mathcal{K}_{\mathcal{L}}L(t) + 2\mathcal{K}_{\mathcal{M}} - \frac{A_2G_2\sum_m\Lambda_m}{\alpha_2}\right)$$
$$-k_{2m}(0)\left[\mathcal{K}_{\mathcal{C}}\mathcal{C}_1(0)(\psi_{\mathcal{C}}+\delta_2)e^{\psi_{\mathcal{C}}t} + \mathcal{K}_{\mathcal{L}}\left(L(0) - l(0)\frac{n}{n+\kappa}\right)(\kappa-\delta_2)e^{-\kappa t}\right.$$
$$\left.-\mathcal{K}_{\mathcal{L}}l(0)\frac{n}{n+\kappa}(n+\delta_2)e^{nt}\right] + (1+\tau_c)\mu_1 - (1+\tau_c)(1+\frac{\eta_2}{\eta_1})e^{\psi_{\mathcal{C}}t}\mathcal{C}_{1m}(0)$$
$$-\frac{\mu_2[(1-\tau_b)r_b+\delta_2]}{\alpha_2} + T(t)\ Ind_m + pA_2G_2\ \Lambda_m$$

Substituting for $\mathcal{C}_1(t)$ we obtain

$$\dot{B}_m(t) = (1-\tau_b)r_bB_m(t) + \left[(1-\tau_w)w_1 - \frac{\lambda^2{}_{hb}-1}{2\theta}\right]\left[e^{(n+\gamma)t}\mathcal{A}_m - e^{nt}\mathcal{B}_m - e^{-\kappa t}\mathcal{D}_m\right]$$
$$+\left[(1-\tau_k)r_1 - \frac{\lambda^2{}_{k_1b}-1}{2\pi}\right]K_{1m}(0)e^{\psi_{k_1}t}$$
$$+\left[w_2 - \frac{2\lambda_{hb}}{S'(\beta)}\left(1+\frac{S(\beta)\lambda_{hb}}{S'(\beta)}\right)\right]\ \left[e^{-\kappa t}\mathcal{E}_m + e^{nt}\mathcal{F}_m\right] + r_2k_{2m}(0)\left[\mathcal{K}_{\mathcal{C}}\mathcal{C}_1(0)e^{\psi_{\mathcal{C}}t}\right.$$
$$\left.-\mathcal{K}_{\mathcal{L}}\left(L(0) - l(0)\frac{n}{n+\kappa}\right)e^{-\kappa t} - \mathcal{K}_{\mathcal{L}}l(0)\frac{n}{n+\kappa}e^{nt} + 2\mathcal{K}_{\mathcal{M}} - \frac{A_2G_2\sum_m\Lambda_m}{\alpha_2}\right]$$
$$-k_{2m}(0)\left[\mathcal{K}_{\mathcal{C}}\mathcal{C}_1(0)(\psi_{\mathcal{C}}+\delta_2)e^{\psi_{\mathcal{C}}t} + \mathcal{K}_{\mathcal{L}}\left(L(0) - l(0)\frac{n}{n+\kappa}\right)(\kappa-\delta_2)e^{-\kappa t}\right.$$
$$\left.-\mathcal{K}_{\mathcal{L}}l(0)\frac{n}{n+\kappa}(n+\delta_2)e^{nt}\right] + (1+\tau_c)\mu_1 - (1+\tau_c)(1+\frac{\eta_2}{\eta_1})e^{\psi_{\mathcal{C}}t}\mathcal{C}_{1m}(0)$$
$$-\frac{\mu_2[(1-\tau_b)r_b+\delta_2]}{\alpha_2} + T(t)\ Ind_m + pA_2G_2\ \Lambda_m$$

Simplifying we obtain the following, which is employed to obtain (79).

$$\begin{aligned}
\dot{B}_m(t) &= (1-\tau_b)r_b B_m(t) + \left[(1-\tau_w)w_1 - \frac{\lambda^2{}_{hb}-1}{2\theta}\right]\left[e^{(n+\gamma)t}\mathcal{A}_m - e^{nt}\mathcal{B}_m - e^{-\kappa t}\mathcal{D}_m\right] \\
&+\left[(1-\tau_k)r_1 - \frac{\lambda^2{}_{k_1 b}-1}{2\pi}\right]K_{1m}(0)e^{\psi_{k_1}t} \\
&+\left[w_2 - \frac{2\lambda_{hb}}{S'(\beta)}\left(1+\frac{S(\beta)\lambda_{hb}}{S'(\beta)}\right)\right]\ \left[e^{-\kappa t}\mathcal{E}_m + e^{nt}\mathcal{F}_m\right] \\
&+r_2 k_{2m}(0)\left[\mathcal{K}_{\mathcal{C}}\mathcal{C}_1(0)e^{\psi_{\mathcal{C}}t} - \mathcal{K}_{\mathcal{L}}e^{-\kappa t}\mathcal{E} - \mathcal{K}_{\mathcal{L}}e^{nt}\mathcal{F} + 2\mathcal{K}_{\mathcal{M}} - \frac{A_2 G_2 \sum_m \Lambda_m}{\alpha_2}\right] \\
&-k_{2m}(0)\left[\mathcal{K}_{\mathcal{C}}\mathcal{C}_1(0)(\psi_{\mathcal{C}}+\delta_2)e^{\psi_{\mathcal{C}}t} + \mathcal{K}_{\mathcal{L}}(\kappa-\delta_2)e^{-\kappa t}\mathcal{E} - \mathcal{K}_{\mathcal{L}}(n+\delta_2)e^{nt}\mathcal{F}\right] \\
&+(1+\tau_c)\mu_1 - (1+\tau_c)(1+\frac{\eta_2}{\eta_1})e^{\psi_{\mathcal{C}}t}\mathcal{C}_{1m}(0) - \frac{\mu_2[(1-\tau_b)r_b+\delta_2]}{\alpha_2} \\
&+pA_2G_2\ \Lambda_m + T(t)\ Ind_m
\end{aligned}$$

B Derivation of intertemporal welfare of household m.

$$\begin{aligned}
u_m &= \int_0^\infty e^{-\rho t}U\left(C_{1m}(t), C_{2m}(t), G_c(t)\right)e^{-\rho t}dt \\
&= \int_0^\infty \frac{e^{-\rho t}}{1-\epsilon}\left[\chi_1^{\eta_2}\mathcal{C}_{1m}(t)G_c(t)\right]^{1-\epsilon}dt \\
&= \int_0^\infty \frac{e^{-\rho t}}{1-\epsilon}\left[\left(\frac{\eta_2(1+\tau_c)}{\eta_1 p}\right)^{\eta_2}\mathcal{C}_{1m}(0)e^{\psi_{\mathcal{C}}t}\ g_c e^{\psi_{\mathcal{C}}t}\left(\sum_m \mathcal{C}_{1m}(0)\right)\right]^{1-\epsilon}dt \\
&= \left(\frac{1}{1-\epsilon}\right)\left(\frac{\eta_2(1+\tau_c)}{\eta_1 p}\right)^{\eta_2(1-\epsilon)}\left[\mathcal{C}_{1m}(0)\ g_c\left(\sum_m \mathcal{C}_{1m}(0)\right)\right]^{1-\epsilon}\int_0^\infty e^{-\rho t}e^{2\psi_{\mathcal{C}}(1-\epsilon)t}dt \\
&= \left(\frac{1}{1-\epsilon}\right)\left(\frac{\eta_2(1+\tau_c)}{\eta_1}\right)^{\eta_2(1-\epsilon)}\left[\mathcal{C}_{1m}(0)\ g_c\left(\sum_m \mathcal{C}_{1m}(0)\right)\right]^{1-\epsilon}\int_0^\infty e^{[2\psi_{\mathcal{C}}(1-\epsilon)-\rho]t}dt \\
&= \left(\frac{1}{1-\epsilon}\right)\left(\frac{\eta_2(1+\tau_c)}{\eta_1 p}\right)^{\eta_2(1-\epsilon)}\left[\mathcal{C}_{1m}(0)\ g_c\left(\sum_m \mathcal{C}_{1m}(0)\right)\right]^{1-\epsilon} \\
&\qquad\qquad \left(\frac{1}{2\psi_{\mathcal{C}}(1-\epsilon)-\rho}\right)\left[e^{[2\psi_{\mathcal{C}}(1-\epsilon)-\rho]s}\right]_0^\infty \\
&= \left(\frac{1}{1-\epsilon}\right)\left(\frac{\eta_2(1+\tau_c)}{\eta_1 p}\right)^{\eta_2(1-\epsilon)}\left[\mathcal{C}_{1m}(0)\ g_c\left(\sum_m \mathcal{C}_{1m}(0)\right)\right]^{1-\epsilon}\left(\frac{1}{\rho-2\psi_{\mathcal{C}}(1-\epsilon)}\right) \\
&= \left(\frac{1}{1-\epsilon}\right)\left(\frac{\eta_2(1+\tau_c)}{\eta_1 p}\right)^{\eta_2(1-\epsilon)}(g_c\mathcal{C}_1(0))^{1-\epsilon}\left(\frac{1}{\rho-2\psi_{\mathcal{C}}(1-\epsilon)}\right)\mathcal{C}_{1m}(0)^{1-\epsilon} \\
&=: \mathcal{U}\left(\mathcal{C}_{1m}(0), \mathcal{C}_1(0), \widehat{\mathcal{P}},\right)
\end{aligned}$$

Here we have employed the facts that, in equilibrium, we have (See Sections 5 and 30):

$$G_c(t) = g_c \mathcal{C}_1(t) = g_c \left(\sum_m \mathcal{C}_{1m}(t) \right) = g_c \left(\sum_m \mathcal{C}_{1m}(0) \right) e^{\psi_{\mathcal{C}} t}$$

and under the maintained Stone Geary preference structure, we have (See Section 6.1):

$$U\left(C_{1m}, C_{2m}, G_c\right) = \frac{1}{1-\epsilon} c\left(C_{1m}, C_{2m}\right)^{1-\epsilon} G_c^{1-\epsilon} = \frac{1}{1-\epsilon} \left(\mathcal{C}_{1m}^{\eta_1} \mathcal{C}_{2m}^{\eta_2} G_c\right)^{1-\epsilon} = \frac{1}{1-\epsilon} \left(\chi_1^{\eta_2} \mathcal{C}_{1m} G_c\right)^{1-\epsilon}.$$

C Deriving the transversality condition for $B_2(t)$.

Recall that $(1-\tau_b) r_b = \mathcal{Z}_1$. Hence, solving the differential equation (91) we get

$$\begin{aligned}
B_2(t) e^{-\mathcal{Z}_1 t} \;&=\; B_2(0) + \int_0^t \Big[\mathcal{Z}_2 \mathcal{A}_2 e^{(n+\gamma)t} + \mathcal{Z}_{5a2} e^{nt} + \mathcal{Z}_4 \mathcal{F} k_{22}(0) e^{nt} + \mathcal{Z}_{5b2} e^{-\kappa t} + \mathcal{Z}_6 \mathcal{E} k_{22}(0) e^{-\kappa t} \\
&+ \mathcal{Z}_7 K_{12}(0) e^{\psi_{k_1} t} + \mathcal{Z}_8 \mathcal{C}_1(0) k_{22}(0) e^{\psi_{\mathcal{C}} t} + \mathcal{Z}_9 \mathcal{C}_{12}(0) e^{\psi_{\mathcal{C}} t} + \left[\mathcal{Z}_{10} - \frac{r_2 A_2 G_2 \sum_m \Lambda_m}{\alpha_2} \right] \frac{k_{22}(0)}{\mathcal{Z}_1} \\
&+ \tau_b r_b B(t) + \tau_k r_1 K_1(0) e^{\psi_{k1} t} + \tau_w w_1 \left(\mathcal{A} e^{(n+\gamma)t} - \mathcal{B} e^{nt} + \mathcal{D} \kappa e^{-\kappa t} \right) + (\tau_c - g_c) \mathcal{C}_1(0) e^{\psi_{\mathcal{C}} t} \\
&+ \mathcal{Z}_{11} - G_1(1 - \mathcal{I} A_1) - p G_2 (1 - \Lambda_2 A_2) + 2\tau_c \mu_1 \Big] e^{-\mathcal{Z}_1 s} ds \qquad (104)
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow \quad \tau_b r_b \int_0^t B(s) e^{-\mathcal{Z}_1 s} ds \;&=\; \Theta[e^{\tau_b r_b} - 1] + \frac{\tau_b r_b \Gamma_1}{(n+\gamma-r_b)(n+\gamma-\mathcal{Z}_1)} [e^{(n+\gamma-\mathcal{Z}_1)t} - 1] \\
&- \frac{\tau_b r_b \Gamma_2}{(n-r_b)(n-\mathcal{Z}_1)} [e^{(n-\mathcal{Z}_1)t} - 1] - \frac{\tau_b r_b \Gamma_3}{(\kappa+r_b)(\kappa+\mathcal{Z}_1)} [e^{-(\kappa+\mathcal{Z}_1)t} - 1] \\
&+ \frac{\tau_b r_b \Gamma_4}{(\psi_{k_1} - r_b)(\psi_{k_1} - \mathcal{Z}_1)} [e^{(\psi_{k_1} - \mathcal{Z}_1)t} - 1] - \frac{\tau_b r_b \mathcal{C}_1(0) \Gamma_5}{(\psi_{\mathcal{C}} - r_b)(\psi_{\mathcal{C}} - \mathcal{Z}_1)} [e^{(\psi_{\mathcal{C}} - \mathcal{Z}_1)t} - 1] \\
&+ \tau_b \left(\frac{2\mu_1 + G_1(\mathcal{I} A_1 - 1) + p G_2 (\sum_m \Lambda_m A_2 - 1)}{\mathcal{Z}_1} \right) [e^{-\mathcal{Z}_1 t} - 1]
\end{aligned}$$

Plugging this into (104) we obtain the following. Conditions ensuring TVC for $B_2(t)$ follow.

$$\begin{aligned}
B_2(t) e^{-\mathcal{Z}_1 t} &= B_2(0) + \frac{\mathcal{Z}_2 \mathcal{A}_2}{n+\gamma-\mathcal{Z}_1} [e^{(n+\gamma-\mathcal{Z}_1)t} - 1] + \frac{\mathcal{Z}_{5a2}}{n-\mathcal{Z}_1} [e^{(n-\mathcal{Z}_1)t} - 1] + \frac{\mathcal{Z}_4 \mathcal{F} k_{22}(0)}{n-\mathcal{Z}_1} [e^{(n-\mathcal{Z}_1)t} - 1] \\
&- \frac{\mathcal{Z}_{5b2}}{\kappa + r_b} [e^{(-(\kappa+r_b))t} - 1] - \frac{\mathcal{Z}_6 \mathcal{E} k_{22}(0)}{\kappa + r_b} [e^{(-(\kappa+r_b))t} - 1] + \frac{\mathcal{Z}_7 K_{12}(0)}{\psi_{k_1} - \mathcal{Z}_1} [e^{(\psi_{k_1} - \mathcal{Z}_1)t} - 1] \\
&+ \frac{\mathcal{Z}_8 k_{22}(0) \mathcal{C}_1(0)}{\psi_{\mathcal{C}} - \mathcal{Z}_1} [e^{(\psi_{\mathcal{C}} - \mathcal{Z}_1)t} - 1] + \frac{\mathcal{Z}_9 \mathcal{C}_{12}(0)}{\psi_{\mathcal{C}} - \mathcal{Z}_1} [e^{(\psi_{\mathcal{C}} - \mathcal{Z}_1)t} - 1] - \left[\mathcal{Z}_{10} - \frac{r_2 A_2 G_2 \sum_m \Lambda_m}{\alpha_2} \right] \frac{k_{22}(0)}{\mathcal{Z}_1} [e^{(-\mathcal{Z}_1)t} - 1] \\
&+ \frac{\tau_k r_1 K_1(0)}{\psi_{k_1} - \mathcal{Z}_1} [e^{(\psi_{k_1} - \mathcal{Z}_1)t} - 1] + \frac{\tau_w w_1 \mathcal{A}}{n+\gamma-\mathcal{Z}_1} [e^{(n+\gamma-\mathcal{Z}_1)t} - 1] \\
&- \frac{\tau_w w_1 \mathcal{B}}{n - \mathcal{Z}_1} [e^{(n-\mathcal{Z}_1)t} - 1] - \frac{\tau_w w_1 \kappa \mathcal{D}}{\kappa + r_b} [e^{-(\kappa+r_b)t} - 1] + \frac{(\tau_c - g_c) \mathcal{C}_1(0)}{\psi_{\mathcal{C}} - \mathcal{Z}_1} [e^{(\psi_{\mathcal{C}} - \mathcal{Z}_1)t} - 1] \\
&- \frac{(\mathcal{Z}_{11} - G_1(1 - \mathcal{I} A_1) - p G_2 (1 - \Lambda_2 A_2) + 2\tau_c \mu_1)}{r_b} [e^{(-\mathcal{Z}_1)t} - 1] + \tau_b r_b \int_0^t B(s) e^{-\mathcal{Z}_1 s} ds
\end{aligned}$$